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A Two-Dimensional Measure of Polarization

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INTERNATIONAL FOOD POLICY RESEARCH INSTITUTE

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ABSTRACT

The link between economic distribution and social conflict—and the notion that this link arises from individuals' sense of identification with those similar to them and their feelings of alienation from individuals with different characteristics—has spurred a literature on polarization, a concept distinct from inequality. This literature, with few exceptions, has nearly exclusively focused on polarization along one (i.e., economic) dimension, despite ample evidence that identification and alienation are often formulated along noneconomic attributes. This paper extends previous work by presenting and discussing a measure of polarization that allows analysis of the distribution of society along two dimensions—an economic variable (e.g., income) and an immutable variable with social significance (e.g., skin color). The measure is discussed in light of four axioms that specify the types of distributional changes that should reasonably translate into a higher degree of socioeconomic polarization. Applying the measure to a family of functions that can represent both unimodal and bimodal population distributions, the measure satisfies the four axioms—briefly summarized as a shrinking of the middle class, greater concentration of the population around poles, greater distance between the poles, and higher correlation between the two variables—under certain parametric restrictions. Unlike the existing studies, which explore multidimensional polarization, we propose a polarization measure that treats the social attribute as continuous (and hence with ordinal properties), thus being able to capture both identification and alienation in social and economic terms.

Keywords: two-dimensional polarisation; socio-economic polarisation; alienation; identity; distribution

1. INTRODUCTION

A Two-Dimensional Measure of Polarization

A body of work drawing from endogenous growth theory and political economy models makes the link between inequality and growth by modeling the cost of distributional conflict that is likely to emerge in highly unequal economies. However, recent literature has argued that in analyzing the role of distribution in the emergence of conflictual situations, the concept of inequality misses important features of wealth and income distribution that may tend to encourage social tension.[1] For example, Esteban and Ray (1999) develop a behavioral framework showing that the types of distribution that magnify social conflict are largely characterized by greater degrees of polarization. Other studies, such as Esteban and Ray (1994) and Duclos, Esteban, and Ray (2004), more explicitly build a measure of a population's degree of economic polarization, with an underlying framework that identifies distributional forms that are conducive to the emergence of conflictual relations in society. In particular, societies may be more prone to conflict when the population is concentrated into distinct, tightly delineated subgroups that are socially distant from one another.

The idea itself is not new—an older literature on class structure forwarded the notion that the existence of two distinct economic classes, one significantly better endowed than the other, would help generate strong identification of people with their class and would be the foundation for the emergence of conflict and revolution. This suggests, then, that conflict is more likely to arise in certain social structures through the effect these structures have on people's identification with their group and a sense of alienation from others—or, in short, through the way that distribution affects identity. The measure of polarization in Esteban and Ray (1994) seeks to capture these conflict-inducing features of distribution.

However, Esteban and Ray's as well as several other polarization measures account only for the distribution of people over one economic variable, such as wealth or income.[2] Yet there is ample evidence that identities are rarely formed exclusively—and often not even predominantly—on the basis of people's economic positions; they are also formed on noneconomic variables, such as ethnicity, religious affiliation, and so forth.[3] In addition, intergroup and intragroup inequality may interface in ways that instigate conflict in qualitatively and quantitatively very different ways, depending on whether the group is defined in economic terms, implying that individuals can potentially enter or exit such groups, or whether they are defined in social and relatively immutable terms (e.g., ethnicity). See, for example, Robinson (2001) for a careful discussion of this latter distinction.

In fact, Esteban and Ray (2006) identify features of this interplay in a model of ethnic conflict. This more nuanced framework of polarization, inequality, and conflict, which explicitly accounts for the role of ethnicity as a noneconomic characteristic, counters some of their earlier findings. In this recent study, for example, greater within-group inequality, as well as the concomitant increase in economic polarization, no longer necessarily leads to greater probability of social conflict, once the group is no longer defined over the economic variable (as was the case in Esteban and Ray [1994]—henceforth, ER) but instead over a social variable, such as religion.

One straightforward way to move from ER's one-dimensional measure to a two-dimensional measure of polarization has been applied in D'Ambrosio (2001), which treats geographical areas, rather than income classes, as the relevant group over which identification (and intergroup alienation) is defined.

[1] Examples of such studies on inequality and conflict include Persson and Tabellini (1994), Alesina and Rodrik (1994), and Keefer and Knack (2002). In most such studies, "conflict" emerges only in the narrow sense of the word, in that heterogeneous agents differ in terms of which policy parameters they find optimal, leading to redistributional policies that hamper growth.

[2] Other recent studies providing a formal representation of polarization in one-dimensional space include Wolfson (1994), Beach et al. (1997), Gradín (2002), Montalvo and Reynal-Querol (2003), Anderson (2004), and Chakravarty, Majumder, and Roy (2007). Esteban and Ray (2005) discuss and categorize various one-dimensional polarization measures.

[3] The literature on identity and conflict abounds. A few examples are Dahrendorf (1959) on economic class, Horowitz (1985) on ethnicity, Eriksen (2001) on ethnicity and culture, Brewer (2001) on the dynamic between intragroup identification and intergroup hostility, and Schlee (2004) on the construction of identity in contexts of conflict.

This consideration of a second noneconomic dimension (in this example, region) introduces—unlike ER, but similar to Duclos et al. (2004)—a nondegenerate density of income within a group. D'Ambrosio (2001) then separately analyzes this within-group (i.e., within-region) inequality over time. After having empirically established no significant qualitative differences between standard inequality measures and unidimensional polarization measures in their application of these measures to China, Zhang and Kanbur (2001) were motivated to define the group not over income classes, but over geographical variables (rural-urban and alternatively coastal-inland), similar to D'Ambrosio (2001). In this framework, the polarization measure is the ratio of intergroup inequality to intragroup inequality.

The literature motivates further attempts to develop a formal approach to the measure of polarization that can take into account social dimensions and that can capture the interplay between individuals' material welfare and their social characteristics. This paper presents and discusses a measure of polarization that would allow analysis of the distribution of society along these two dimensions. Unlike the studies discussed previously, however, the two-dimensional space considered in this paper is fully continuous, as opposed to being continuous in only the economic dimension and discrete in the social dimension. To the author's knowledge, there has not been a prior attempt at formalizing a multidimensional and bicontinuous measure of polarization. The use of fully continuous space not only has technical-analytical implications—of both a more challenging as well as a more facilitating nature—but also implies a different conceptual understanding of how social forces may play a role in defining *polarization*. This will be discussed in more detail later in the paper, including in the final remarks of the last section.

Although research on multidimensional polarization is just beginning to emerge and is as yet very sparse, there does exist an interesting body of work on multidimensional inequality measures, with Tsui (1995) being among the first to provide a formal axiomatic approach to such a measure. (See also Tsui [1999]; Amartya Sen's body of work [including Sen (1973) and Sen (1992)] on multifaceted welfare and welfare distribution, which provided the conceptual motivation for subsequent measurement development; and, for a recent summary of multidimensional inequality indices, Diez et al. [2007].) However, what distinguishes the literature in a rather fundamental way from the enterprise of this paper (as well as that of the other studies previously discussed, e.g., Zhang and Kanbur [2001], D'Ambrosio [2001], and Robinson [2001]) is that the inequality studies are concerned with multiple attributes of individuals, all of which reflect a different dimension of well-being. For example, measuring income inequality alone may overstate the extent of inequitable welfare in a society, especially if income is highly unequally distributed while access to services, such as health care or electricity, is strongly egalitarian. In this example, a multidimensional inequality measure that accounts for access to health and energy services can more adequately capture the distribution of welfare in the society than can a univariate, income-based measure alone.

In contrast, the endeavor in this paper, as in the other papers cited, is quite different: The multidimensionality is not so much motivated by seeking to account for different measures of economic well-being, but rather by seeking to explore how the distribution of an economic variable, such as income, along a noneconomic (and not intrinsically welfare-related) dimension may be measured and how that may matter. Furthermore, in this paper, as in most of the other examples, the social variable is treated as immutable, in the sense that individuals' social attributes do not change over time. In contrast, all attributes in the models of the analogous inequality literature are, in principle, subject to change for a given observation (e.g., individual).

Despite the different motivations for introducing multidimensionality into the distribution measures in the polarization and inequality literatures, the latter literature may be able to offer technical contributions to further develop (continuous) two-dimensional polarization measures. This study does not use the approaches established by Tsui (1995, 1999) and other authors as a point of departure; instead, it uses the one-dimensional polarization measures by ER and Duclos et al. (2004). Future studies, however, may want to explore alternative multivariate polarization measures that draw on the analogous inequality literature.

Section 2 briefly recapitulates the univariate polarization measure as developed in ER and Duclos et al. (2004) and outlines the framework underlying the polarization measure. Section 3 describes, in intuitive terms, criteria that a polarization measure should satisfy. These axioms are represented formally in Section 4. Section 5 formulates a polarization measure in two dimensions and describes its basic characteristics, while Section 6 assesses this measure in light of the polarization axioms. Section 7 compares the measure with its univariate analogue; the final section concludes and proposes areas for future research.

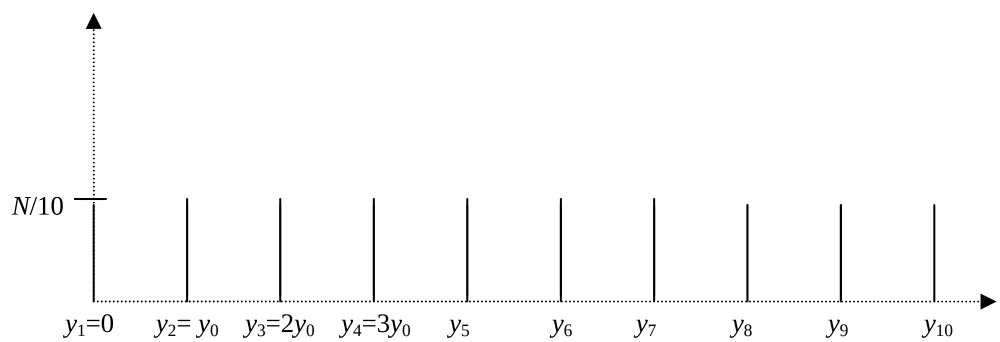
2. A ONE-DIMENSIONAL MEASURE OF POLARIZATION

The formal, explicit modeling of polarization is new to economics, though other social sciences, especially sociology and political science, took up this concept of distribution early on. The first serious effort at formalizing the concept of polarization in the field of economics was launched simultaneously by ER and Wolfson (1994). Wolfson's approach to a polarization measure directly used the Lorenz curve model as a starting point. ER, on the other hand, offered an axiomatic approach to developing a polarization measure and founded it on a framework of interpersonal relations that are assumed to be related, in their aggregate, to the likelihood of social conflict. ER's framework is briefly explained in this section.

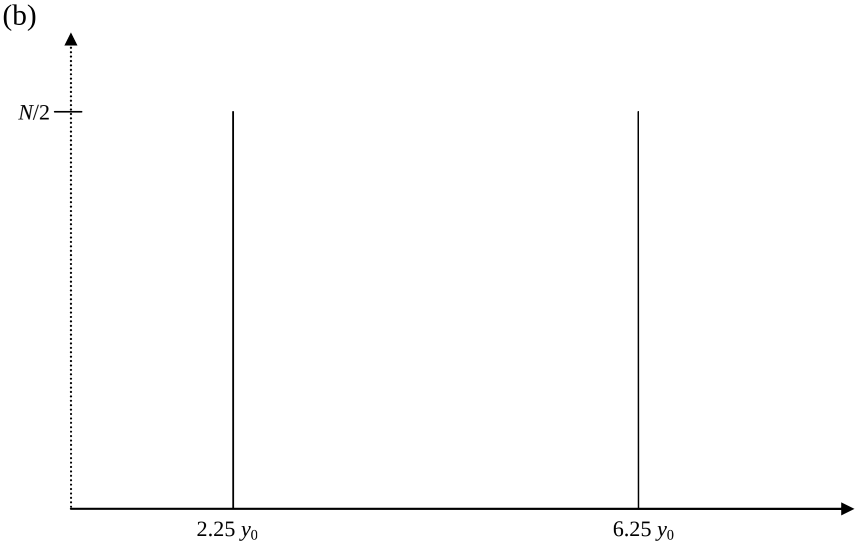
First, to help develop the basic intuition behind the polarization concept, which can easily be confused with inequality, a simple case will illustrate why polarization—as a description of the way a certain attribute is distributed among individuals—is distinct from the inequality descriptor. Consider some discrete distribution of N individuals over m equally spaced wealth classes y_0 apart, with the lowest wealth class owning 0, in such a way that there are an equal number (N/m) of people in each class (Figure 1a displays the case m = 10).

Figure 1. Comparison of inequality and polarization

(a)



(b)



Changing this distribution so that half the population now owns wealth level ¼ y_m and the other half moves to the position ¾ y_m (see Figure 1b) reduces wealth inequality. The Gini coefficient for the first distribution is

$$G = \frac{1}{2N^2\bar{y}}\sum_{i=1}^{N}\sum_{j=1}^{N}| y_i - y_j | = \frac{1}{2N^2((m-1)y_0/2)}\sum_{k=1}^{m-1}(m-k)(ky_0 \cdot (N/m)^2) \qquad (1)$$

so,

$$G = \frac{1}{m^2(m-1)}\sum_{k=1}^{m-1}(m-k)k \qquad (2)$$

and the second distribution always equals G = ⅛ for any $m > 2$ and any N. It is easy to see from comparing (2) with G = ⅛ that the two-class distribution is less unequal than is the distribution with multiple classes.[4]

However, by some intuitive sense of the term *polarization*, one would not expect, from inspecting Figure 1, that the two-class distribution is also less polarized. On the contrary, a measure of polarization should reveal the second distribution as being the more polarized one. ER make explicit the intuition that underlies such polarization ranking by formulating three axioms. A measure of polarization should increase when a distribution changes in such a way that (1) small proximate groups merge to one large group (increased homogeneity within a broadly defined income class), (2) two distinct groups move farther apart from each other (increased heterogeneity between income classes), and (3) in a three-group system, part or all of the middle group becomes absorbed by the other two (shrinking middle class).

Because the polarization measure proposed in this paper is a direct extension of ER and Duclos et al. (2004), it is useful to first briefly summarize the conceptual framework underlying this measure. Consider some attribute of individuals in a population that may, in part, inform individuals' identities. We continue the example of economic class represented by wealth, as used in the discussion of Figure 1. Polarization as a feature of overall distribution of wealth has as its core element the degree of alienation of each individual in society toward each other person in that society. Two elements influence the sentiment of alienation of some person toward another: The first is the distance between the two individuals. In the context of wealth distribution, this is the wealth gap between person *a* and some other person *b*. The second is the strength of self-identification, or the extent to which the individual feels strongly about his or her class identity.

Someone whose economic class is very distant from another's and who strongly identifies with his or her wealth group is considered to be highly alienated from the other individual. Formally, the degree of effective alienation of *a* toward *b* is defined as

$$T(y_a, y_b) = t(\phi(y_a, y_b) \cdot J(y_a)) \qquad (3)$$

where ϕ measures wealth distance, $J(y_a)$ measures the strength of *a*'s self-identification, and $t(\,)$ is a monotonically increasing function. Wealth distance is the absolute value of the difference between *a*'s and *b*'s endowments,

$$\phi(y_a, y_b) = |y_a - y_b| \qquad (4)$$

The identification function is straightforwardly defined as

$$J(y_a) = n(y_a)^\alpha,\ \alpha > 0 \qquad (5)$$

[4] This is the case for $m < 40$.

which measures the strength of the individual's identity based on how many people, n, are like that individual—that is, how many also have endowments y_a. In this sense, the framework relates the strength of an individual's self-identification to the size of his or her class.

With a simple function $t(z) = z$, the alienation function becomes

$$T(y_a, y_b) = |y_a - y_b| \cdot n(y_a)^{\alpha}$$

The degree of overall polarization in a discrete distribution is then the sum of all pairwise alienations—that is,

$$P_D^y = K\sum_{i=1}^{m}\sum_{j=1}^{m}\left(| y_i - y_j | \cdot n(y_i)^{\alpha}\right)\cdot n(y_i) \cdot n(y_j) = K\sum_{i=1}^{m}\sum_{j=1}^{m}| y_i - y_j | \cdot n(y_i)^{1+\alpha} \cdot n(y_j)$$

ER interpret α as the polarization sensitivity of the measure, showing that when α is as low as 0, the measure is directly proportional to the Gini measure of inequality (compare with (1)). ER also show that P_D^y satisfies the above mentioned axioms. Duclos et al. (2004) extend the measure into a continuous space, defining it as

$$P^y = \int\int | y_a - y_b | f(y_a)^{1+\alpha} f(y_b)\, dy_a\, dy_b \tag{6}$$

Although both P_D^y and P^y capture the degree of polarization in society based on individuals' alienation from others in terms of their economic class identification and wealth distances, these measures fail to account for how social forces may influence identity and therefore polarization. Specifically, and as Stewart (2002) emphasizes, both social conflict and arrested development are more likely to exist in settings where material endowment correlates strongly along ethnic, religious, or other socially relevant characteristics. Mogues and Carter (2005) go into greater detail on this phenomenon, linking economic outcomes with socioeconomic polarization. These and other studies on the interplay of multiple group identities in conflict generation, along with the very foundation of the polarization concept on distributions with higher conflict potential, motivate an examination of the degree of polarization along social *and* economic attributes.

3. A SET OF AXIOMS FOR THE DEGREE OF POLARIZATION IN TWO DIMENSIONS

This section proposes, in informal terms, a set of population characteristics that can intuitively be seen as associated with the concept of polarization. In so doing, it synthesizes from the literature the most commonly expressed attributes seen as generating a process of polarization. Section 4 expresses these criteria more formally as axioms and proposes a two-dimensional polarization measure that satisfies these axioms.

Shrinking of the Middle Class

In the economics and other social sciences literature, polarization is most frequently perceived as a process of the hollowing out, or the dwindling away, of the middle class. Several studies of economic distribution in the United States (Horrigan and Haugen 1988; Levy and Murnane 1992; Beach, Chaykowski, and Slotsve 1997; Duncan, Smeeding, and Rogers 1993) describe the expansion of both the groups of upper and lower earners at the expense of the size of the middle class. As also previously discussed, several of these studies emphasize that the erosion of the middle class need not result in greater inequality and that, even where it does, the increase in inequality is not usually the best way to represent a distributional change of this sort. Thus, the concept of polarization ought to address this shortcoming in representing and measuring this economically and socially important phenomenon.

Population Concentration around Poles

An often related, yet distinct, evolution in a distribution is the concentration of individuals or other entities in certain spheres of a spectrum of income, wealth, and so forth. For example, an economy may evolve in such a way that larger masses of people find themselves within more narrowly delimited income classes, whereas before individuals may have been dispersed more widely along the income spectrum. The phenomenon of greater concentration around poles has been used for several decades in the political sciences literature, often with reference to a political and economic ideological spectrum—for example, when more countries become clustered in tight power coalitions with little interaction between nation groups (Deutsch and Singer 1964; Rapkin, Thompson, and Christopherson 1979). The emergence of conflict has been attributed to this distributional evolution in various contexts (in Wright 1965; Burton 1965; Thompson 1986; and Layman and Carsey 2002; to name only a few). In the economic realm, it is insufficient and often inaccurate to describe this as an increase in inequality. Yet we would expect that a measure of polarization should capture such bunching of elements around poles.

Separation of the Poles

Although both the dwindling of the middle class and the concentration around poles in a distribution have been frequently associated with an increase in polarization in studies of economic and noneconomic phenomena, a third, but somewhat less explored, trend is the moving apart of clusters in a distribution. Certainly, what comes to mind with the phrase “the poor are getting poorer and the rich are getting richer” is an increase in economic inequality; indeed, it will usually be the case that a distribution in which the distance between wealth clusters expands is one that is more unequal. In this context, we should expect that a polarization measure and an inequality measure, such as the Gini coefficient, would move in the same direction.

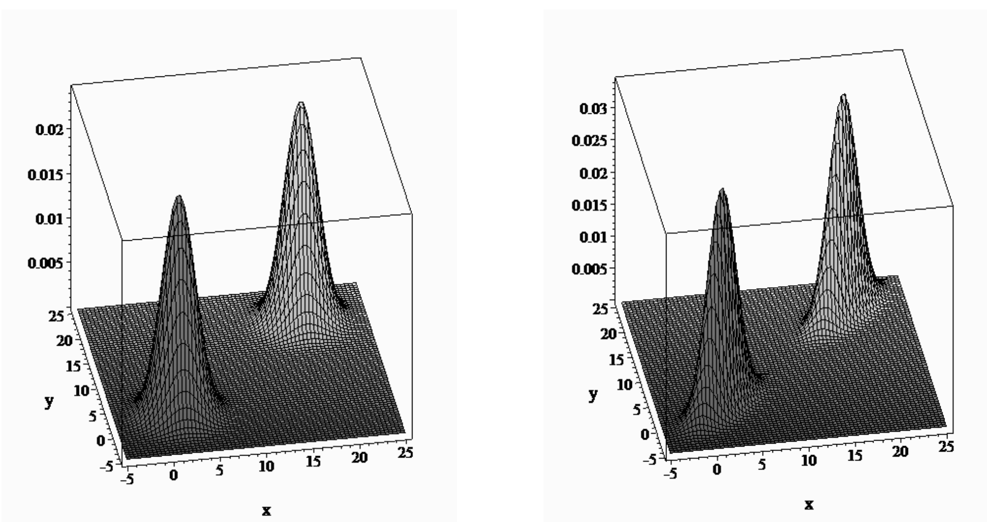
Strong Relationship among Individuals' Attributes

The polarization characteristics suggested so far are applicable to distributions across any number of variables. A characteristic that is particular to multidimensional distributions and that is therefore not as commonly found in references to polarization in the literature (because multivariable characteristics of distribution have rarely been formally examined with respect to polarization and have yet to gain wide currency, even in the inequality literature) is the correlation between two variables. We propose here that a society is to be deemed more polarized if, other things equal, there is a stronger correlation between two variables of interest.

To first give an illustrative example, consider the two distributions in Figure 2. They have in common the number of discernable clusters, the extent to which these clusters are concentrated (to be made more specific below), and the distance between the groups. The only way their distributions differ is in the degree of correlation between the *x* and *y* attributes. A polarization measure in two dimensions should increase with the strength of the relationship between the two variables.

Figure 2. Bimodal distributions with different degrees of correlation

(a) (b)



4. A FORMAL REPRESENTATION OF THE POLARIZATION AXIOMS

Whereas the previous section put forward in general terms a set of criteria for a measure of polarization in a two-dimensional population distribution, this section will make these criteria more specific as formal axioms. But first it describes the functional family of bivariate distributions upon which the polarization axioms will be formulated.

A Specific Family of Population Distributions

For purposes of focus, analysis of the distribution over which a polarization measure is to be defined will be applied to the following functional family with the core properties of a density function (nonnegative support, integrates to 1):

$$g(x, y) = \theta f(x, y; \sigma_{x1}, \sigma_{y1}, \mu_{x1}, \mu_{y1}, \rho_1,) + (1 - \theta) f(x, y; \sigma_{x2}, \sigma_{y2}, \mu_{x2}, \mu_{y2}, \rho_2) \qquad (7)$$

where $f(\,)$ is a bivariate normal distribution with variances, means, and correlation coefficients as given and where $\theta \in [0, 1]$. Special cases of this functional family are distributions with two distinct modes; distributions with one mode; and, as a special case of the latter, the normal distribution when either $\theta = 0$ *or* $\theta = 1$ *or* $f^1(\,) = f^2(\,)$.

Consider the special case of the additive normal distribution (7) in which

$$\theta = ½;\ \sigma = \sigma_{ik} = \sigma_{jm};\ \mu_{xk} = \mu_{yk};\ \mu_{i1} = -\mu_{i2};\ \text{and } \rho_1 = \rho_2 > 0 \qquad (8)$$

where $i, j = \{x, y\}$ and $k, m = \{1, 2\}$.[5] That is, both $g(\,)$ as well as each component distribution $f(\,)$ are symmetric about the two diagonals through their means, and the global mean is the point of origin. The distribution then becomes

$$g_B(x, y, \cdot) = \frac{1}{4\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[b \cdot \left(x^2 + y^2 - 2\rho(xy + \mu^2) + 2\mu^2\right)\right] \cdot \left(\exp(bc) + \exp(-bc)\right) \qquad (9)$$

where $b(\sigma, \rho) = \dfrac{-1}{2\sigma^2(1-\rho^2)}$ and $c(x, y; \mu, \rho) = 2\mu \cdot (x + y)(1 - \rho)$.

In this paper, we will frequently be interested in the case where $g_B(x, y)$ takes on a truly bimodal form. The fact that it is additive in two normal densities with different means does not guarantee that there are two distinct ”hills”; instead, g_B may still be a unimodal distribution. The condition that ensures that the distribution has exactly two maxima depends on all distribution parameters μ, σ, and ρ. Specifically, the condition

$$\frac{1}{1+\rho}\left(\frac{\mu}{\sigma}\right)^2 > w_1 \approx 0.61 \qquad (10)$$

where

$$w_1 = \ln\left(\frac{1}{3}\left(w_2 + \frac{4}{w_2} + 1\right)\right),\ w_2 = \left(19 + 3\sqrt{33}\right)^{1/3}$$

has to hold in order for the density to have two modes. One can say more generally for the case where $\mu_{xk} = a + \mu_{yk}$ and allowing for $\mu_{xk} \neq -\mu_{xm}$, $k, m = \{1, 2\}$ (i.e., where the global mean can be any point and

[5] Throughout, we concern ourselves only with cases of a positive correlation ρ, as negative correlations yield exactly symmetric results.

the normal components are centered on a diagonal through the global mean) that bimodality requires $\frac{1}{1+\rho}\left(\frac{D}{\sigma}\right)^2 > 8w_1$, with D referring to the distance between the two local means and w_1 defined as above (see Appendix A for proof). The following four subsections formalize the axioms stated informally in Section 3, using the functional form laid out above.

Shrinking of the Middle Class

In most discussions about a distributional change that hollows out the middle class—as in the literature referred to in Section 3—there is some explicit or implicit understanding of the income range, wealth group, and so on to which a middle class member would belong. Reference to a shrinking of the middle class, therefore, typically points to a reduction in the population earning incomes or holding wealth within that range. Therefore, in formalizing the notion of the "shrinking middle" in the context of a bimodal distribution (as described above), we measure the mass under the density function over a particular range of variables x and y that can reasonably be taken to constitute the middle range for a given distribution. To have this formulation apply to the notion of a "hollowing out," we consider only truly bimodal distributions—that is, those for which condition (10) holds. Further, let $\theta = ½$, $\sigma = \sigma_{ik} = \sigma_{jm}$, $\mu_{yk} = a + \mu_{xk}$ for some a, and $\mu_{x2} > \mu_{x1}$ and $\rho_1 = \rho_2$ where $i, j = \{x, y\}$ and $k, m = \{1, 2\}$. Let the middle range be defined by some value c so that the middle mass is the volume under the distribution within the limits
$[c_{xl} = (\mu_{x1} + \mu_{x2})/2 - c,\ c_{xu} = (\mu_{x1} + \mu_{x2})/2 + c]$ for x and
$[c_{yl} = a + (\mu_{x1} + \mu_{x2})/2 - c,\ c_{yu} = a + (\mu_{x1} + \mu_{x2})/2 + c]$ for y, or

$$M = \int_{c_{yl}}^{c_{yu}} \int_{c_{xl}}^{c_{xu}} \frac{1}{2}\left(f^1(x,y) + f^2(x,y)\right) dx\, dy = \frac{1}{2}\left(\int_{c_{yl}}^{c_{yu}} \int_{c_{xl}}^{c_{xu}} f^1(x,y)\, dx\, dy + \int_{c_{yl}}^{c_{yu}} \int_{c_{xl}}^{c_{xu}} f^2(x,y)\, dx\, dy \right) \quad (11a)$$

Given that the limits are symmetric to the global mean point, as they ought to be in order to capture the middle region of the distribution, the two integrals of the normal components are equal. Thus,

$$M = \int_{c_{yl}}^{c_{yu}} \int_{c_{xl}}^{c_{xu}} f^i(x,y)\, dx\, dy \quad (11b)$$

where i may stand for 1 or 2. There is no immediate and obvious way of selecting c and therewith the boundaries of the middle region. It would appear natural to limit c to being no larger than $(\mu_{x2} - \mu_{x1})/2$; if it were larger, the middle region would include the modes, which can be understood in this context to be the poles in the distribution. The larger that one chooses $c \in (0, (\mu_{x2} - \mu_{x1})/2)$, the more generous is the definition of what constitutes the middle "class," and vice versa. In addition, as will be shown, the choice of c will determine how a change in certain moment parameters of the distribution can be said to affect the "shrinking middle" criterion of polarization.

The following illustrates the effect of the first moment parameters—namely, the local means. In the above functional form of a bimodal distribution, consider a decrease of μ_{x1} and an increase of μ_{x2} by the same amount. Given that both mean points are on the same diagonal—that is, $\mu_{yk} = a + \mu_{xk}$—this constitutes a movement apart of the normal components along the diagonal. In (11a), we saw that the middle mass can be expressed entirely in terms of only one of these two components. Therefore, the marginal effect of a mean separation need only be examined on one component of the distribution. We arbitrarily choose $f^2(x, y)$, then define

$$h_1 = (x - \mu_{x2})^2 + (y - \mu_{x2} - a)^2 - 2\rho(x - \mu_{x2})(y - \mu_{x2} - a)$$

and

$$h_2 = \frac{h_1}{-2\sigma^2(1-\rho^2)} : \frac{\partial}{\partial \mu_{x2}} f^2(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \cdot (\partial e^{h_2} / \partial \mu)$$

$$= \frac{1}{-4\pi\sigma^4\left(\sqrt{1-\rho^2}\right)^3} \cdot e^{h_2} \left[-2(x-\mu_{x2}) - 2(y-\mu_{x2}-a) - 2\rho(-(y-\mu_{x2}-a)-(x-\mu_{x2}))\right]$$

$$= \frac{1}{2\pi\sigma^4\left(\sqrt{1-\rho^2}\right)^3} \cdot e^{h_2} \left[(x-\mu_{x2}) + (y-(\mu_{x2}+a))(1-\rho)\right]$$

Because $f^2(x, y)$ is the function centered at the larger of the local mean points, the middle region covers an area that is to the left of μ_{x2} and below $\mu_{y2} = a + \mu_{x2}$. For all values of x and y less than μ_{x2} and below $a + \mu_{x2}$, respectively, we see that the function must decrease as μ_{x2} increases. Therefore, the middle mass as defined in (11a) must fall with a greater separation of the local means of a bimodal distribution.

The effect of the local variances on the middle-class criterion of polarization is not quite as unambiguous. As above, consider the marginal effect of a change in the local variance of $f^2(x, y)$:

$$\frac{\partial}{\partial \sigma} f^2(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \left(-\frac{2}{\sigma^3}e^g + \frac{e^{h_2}}{\sigma^2}\frac{-2h_1}{-2(1-\rho^2)\sigma^3}\right)$$

$$= \frac{e^{h_2}}{2\pi\sqrt{1-\rho^2}\,\sigma^3} \cdot \left(\frac{h_1}{(1-\rho^2)\sigma^2} - 2\right)$$

Less dispersion in $f^2(x, y)$ reduces the function's value for those values of x and y for which

$$(x - \mu_{x2})^2 + (y - \mu_{x2} - a)^2 - 2\rho(x - \mu_{x2})(y - \mu_{x2} - a) > 2\sigma^2(1 - \rho^2) \tag{12a}$$

and increases it for those $[x, y]$ for which inequality (12a) is reversed. The left side held at some constant value generates an ellipse centered at $[\mu_{x2}, \mu_{x2} + a]$; so, the inequality describes values of x and y outside of the ellipse. The upper-right corner of the middle region, that is, $[c_{xu}, c_{yu}]$, being outside of the ellipse is sufficient, though not necessary, to suggest that a narrowing of the distribution through decreased variance will lead to greater polarization, as represented by the "shrinking middle class" criterion. Because the density function above the ellipse region increases with σ, there will exist some c_0 such that for all $c > c_0$, the increase of the function above some parts of the middle region will outweigh the shrinking parts above the middle region, so that M may increase overall. For any given distribution, this is less likely to occur if the variance is low, because for any given limit of the middle plane, this plane will overlap less with the ellipse as specified in (12a) when σ is low.

Concentration around Poles

In some sense, the concentration criterion in Section 3 is the continuous distribution equivalent to the discrete criterion that there exist only few groups (see ER, pp. 825–826). To make this concrete, in a discrete distribution with n income classes, it is plausible to expect a distribution to be more polarized when people are clustered in only a few of the n income groups rather than dispersed over many groups. In a continuous context, couching polarization in the language of the number of groups, though not impossible, poses the challenge of determining where a group ends and another begins. Efforts have been made in this respect, especially in empirical analysis (see, e.g., Esteban, Gradín, and Ray 2007).

We formalize the condition on greater concentration around poles as follows: Consider some small value v that represents the volume under a part of the total population distribution. v is chosen such that $v = Vn^{-1}$ where n is some integer greater than 1 and $V \in (0, 1)$. For each point p in xy-space, there exists an area circumscribed by $r = r(v, p)$ such that

$$F(r, p) \equiv \int_{p_y - r}^{p_y + r} \int_{p_x - r}^{p_x + r} f(x, y, \cdot)\ dx\ dy = v$$

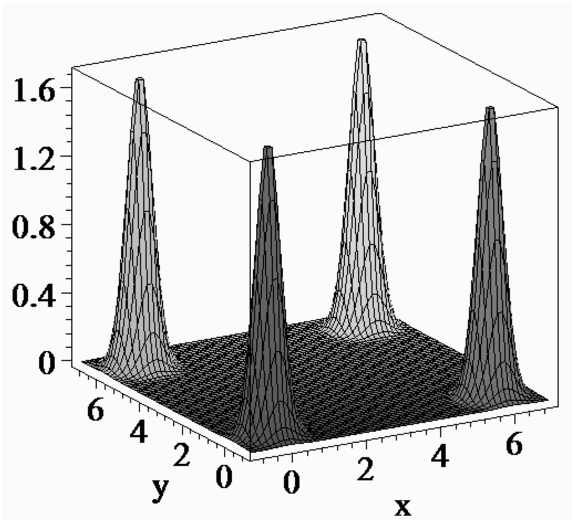
For each given distribution, find $p_1^* = \arg\min\ r(v, p)$, or the vector $[p_x, p_y]$ such that the r-vicinity of p is the smallest area that supports volume v under the density function, and define $r_1^* = (v, p_1^*)$. Next, excluding this area—that is, the plane $\{x \in [p_{x1}^* - r^*, p_{x1}^* + r^*],\ y \in [p_{y1}^* - r^*, p_{y1}^* + r^*]\}$—repeat the process, thus finding $p_2^* = \arg\min\ r(v, p)$, which generates r_2^*, and so on. Comparing across distributions, the distribution with a relatively small value of $\overline{A} \equiv 4\left({}^{1}\!/_{n} \sum_{i=1}^{n} r_i^* \right)^2$ can be said to have a relatively large concentration of the population around poles. This process establishes the degree of concentration for a proportion V of the population. Choosing, for example, $V = 0.5$ and $n = 4$, $\overline{A}(n, V, f(.))$ gives the concentration over the more concentrated population half by determining the average of the four smallest areas that support a mass of ⅛ ($= 0.5 \cdot 4^{-1}$).

Although the coverage V is expressed generally as comprising a positive noncomplete proportion of the population, the measure of concentration will usually be more interesting for intermediate values of V, such as ¼ or ½. For example, if we choose near-complete coverage (V close to 1) and a large number of areas (large n) in a distribution such as that in Figure 3, we would generate a larger value of $\overline{A}$ than what would be expected from inspecting the figure, because the average of the areas would be unduly influenced by the large values of r_i^* for $i > 4$.[6]

Given this definition of concentration, the distributions under consideration become more concentrated as the (local) variances decrease. To illustrate, consider the unimodal distribution centered at the point of origin with $\rho = 0$ and the simple case of $V = ¼$ and $n = 2$. This means we are considering the average of the two smallest areas that each support one-eighth of the population mass. It is clear that the first area is centered at the mean, given that the mean is also the mode. To obtain r that solves $\frac{1}{8} = \int_{-r}^{r}\int_{-r}^{r} f(x, y; \sigma)\ dx\ dy$, note that because $\rho = 0$, $\frac{1}{8} = \int_{-r}^{r} f(x, \sigma)\, dx \cdot \int_{-r}^{r} f(y, \sigma)\, dy = \left(\int_{-r}^{r} f(x, \sigma)\, dx \right)^2$.

[6] It would be interesting to further investigate this measure of concentration by examining whether the ranking of any number of distributions would remain the same for all possible values of V and n, and based on the results, the measure can be further refined. For the specific purpose of formalizing the concentration axiom and for the narrowly defined functional family under consideration, the measure is sufficient.

Figure 3. Four-pole distribution with a high degree of concentration



From the standard normal table, $z_c = 0.46$ for $c = (\sqrt{1/8} + 1)/2 \approx 0.68$ and $r_1^* = z_c \sigma$, which is decreasing in the variance. The second smallest area that supports V is immediately contiguous to the previous area and can be centered horizontally or vertically at the mean point of the distribution. Here the problem is to obtain r that solves $\frac{1}{8} = \int_{z_c\sigma}^{z_c\sigma+2r} \int_{-r}^{r} f(x, y; \sigma)\ dx\ dy$. A closed-form solution for the cumulative distribution function of the normal distribution would be needed to obtain r_2^* explicitly, because the limits differ for integrating over x and y. Figure 4 gives a numeric illustration of the concentration measure $\overline{A} = 2 \cdot \left(r_1^* + r_2^*\right)^2$ for a range of standard deviations.

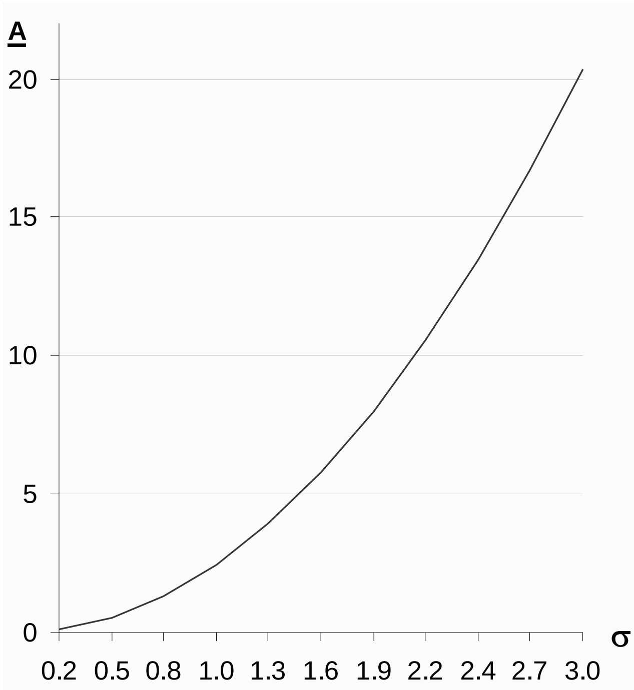
Separation of the Poles

The formal expression of the separation axiom draws on the framework for the concentration axiom: The "poles" are more separate if $D_n = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \| p_i^* - p_j^* \|$, which is the average distance between the central points of greatest concentration. In the case of the symmetric bimodal distribution examined here, the moving apart of modes, or groups within the population, is straightforwardly represented by a greater absolute distance between the local means (which here are equal to the modes), $((\mu_{x1} - \mu_{x2})^2 + (\mu_{y1} - \mu_{y2})^2)^{½}$, holding other parameters constant.

Strength of the Relationship between Attributes Determining the Distribution

This criterion can also be made formal in a simple way by expressing the strength of the relationship between the two dimensions of the joint distribution through their correlation. In the case of additive joint normal distribution, the parameter ρ in the expression of each component function offers itself readily for influencing the relationship between the two variables.

Figure 4. Concentration measure $\overline{A}$ for a bivariate normal distribution with $\rho = 0$



5. A MEASURE OF POLARIZATION IN TWO DIMENSIONS

Having formulated the axioms for the extent of polarization in a continuous population distribution in two dimensions, we now proceed to present a measure and discuss its basic properties. Section 2 stated the one-dimensional measure of ER and Duclos et al. (2004). The two-dimensional polarization measure proposed in this paper is an extension of the latter work (see (6)) and rests on the same alienation framework outlined in Section 2. This modification permits the examination of the extent of societal polarization along an economic and social spectrum, capturing both the degree of polarity in each dimension and how the interrelationship between the dimensions may contribute to overall socioeconomic polarization.

Formulation of the Two-Dimensional Polarization Measure

As the basis for this polarization measure, we assume that there are two dimensions to identity. The first, y, is an economic component (e.g., wealth), and the second component, x, is a characteristic that a particular society deems socially relevant to the construction of identity. Ethnicity, skin color, and maternal language are all examples of such characteristics. For purposes of analysis, we assume that we can characterize both wealth and the social characteristic with a continuous numerical range.

As discussed in the introduction, the use of a continuous social variable in our two-dimensional polarization measure is a departure from the existing literature on this topic, which has been based on discrete concepts such as region or ethnicity. Continuity is, in some respects, technically and conceptually enabling; however, in other respects, it contributes challenges on both fronts. One difference implied by the use of a fully continuous space is that our measure allows for a more natural ordinal understanding of a social variable: Because it is continuous, it immediately follows that there is some sense of a "high" and "low" value on the social scale. This is quite intuitive in many contexts—for example, when race is the salient social feature of individuals, with skin color frequently being the marker in a given society. The ability to conceive of members of society as located on not only an income but also a social scale is germane to the motivation of socioeconomic polarization, as discussed at the outset of this paper and as applied more explicitly in Mogues and Carter (2005). A discretized approach to the social dimension does not necessarily preclude the application of ordinal structure on the social, or noneconomic, variable, but it does preclude taking analytical advantage of the symmetry between both dimensions that arises from bicontinuity. In fact, the two-dimensional measures that we are aware of in the literature do not impose any ordinal values on the noneconomic variable.

Another salient feature in the analytical structure here, distinguishing it from the approaches in the related literature using discrete notions of a social dimension, is that groups are defined over *both* the income and the social dimensions, as opposed to only over the latter. This feature not only has implications for model design, but more fundamentally represents the underlying motivating notion that individuals identify themselves, and experience a degree of alienation toward others, on the basis of both their economic standing and their social status. In a way, this framework unifies features in the ER (and Duclos et al. 2004) type of univariate polarization measures on the one hand, in which groups are defined over income, with studies such as Zhang and Kanbur (2001) on the other, in which the operative group in the measure's intergroup differentiation component is only defined in terms of the social/noneconomic variable.[7]

Clearly, modeling the social dimension as a continuous variable challenges its immediate application to several real-world examples. For example, if gender or religion in a society with only a few dominant religious is the key social feature of interest, a discrete approach appears to be a better representation (though it can be argued that several of these cases may be better represented through a rank-ordered categorical variable than a nonordinal one). However, even in contexts where social groups

[7] Note that in Zhang and Kanbur (2001), for example, the income variable takes a role in the polarization measure by capturing intragroup and intergroup inequality. But what defines the group in the first place is only the noneconomic variable.

appear to be discrete on the surface, there are socially meaningful "shades of gray," making a continuous representation (with appropriate assumptions on the density over this space) applicable. An evident example is the use of "white" and "black" to delineate races versus the use of skin color tones to mark social status. Another example is the reinterpretation of society organized into religious groups (e.g., Christians and Muslims) as individuals lying on the spectrum from fundamentalist Christian to fundamentalist Muslim, with different shades of moderation and secular position in between. We propose that these thoughts mainly suggest that there is in fact space for different types of models and measures in the literature, each serving different empirical contexts as theoretical underpinnings.

Proceeding then with bicontinuity as the approach in our model, consider the range of identity to be given by the plane $[x, y]$. A particular person a's identity is then described by the vector $z_a = [x_a, y_a]$. Let $f(x, y)$ denote the joint distribution of agents in the two-dimensional wealth–social characteristic space.

The identification function and social distance function of (5) and (4), respectively, become

$J(x_a, y_a) = f(x_a, y_a)^{\alpha}$, $\alpha > 0$ and $\phi(z_a, z_b) = \| z_a - z_b \| = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}$, so that the measure is

$$P(\alpha) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}\, f(x_a, y_a)^{1+\alpha}\, f(x_b, y_b)\;\; dx_a\;\, dx_b\;\, dy_a\;\, dy_b \tag{13}$$

The choice of the parameter α determines the importance that the measure P gives to the self-identification component. For higher values of α, alienation of individual a to individual b will be greater, and therefore regions in the distribution with relatively high concentration will contribute more to overall polarization than in a low-α measure.[8] As will be seen in later sections, greater values of α are more likely to satisfy the intuitive criteria detailed in Section 3 for the kinds of changes in a distribution that ought to increase any measure of that distribution's polarization. But first, the remainder of this section describes some basic characteristics of P.

Differentiability and Continuity

P is always differentiable in α and therefore also continuous in α. The former holds because the expression inside the integrals is differentiable with respect to α.

Boundedness

P takes on only nonnegative values. It is an integration over the multiplication of two nonnegative terms: the absolute distance between two points a and b and the density at point a raised to α. It is bound from below by 0. It only takes on the value 0 when the distribution is degenerate, in the sense that all the mass is at one point on the xy-plane, or when the variances are infinitely large.

What is less straightforward to establish is when the measure is bounded from above. P integrates four variables each from minus infinity to plus infinity. Looking at the multiplicative terms in P, we see that as, for example, one of the variables becomes infinitely large, the density components $f_a(\,)^{1+\alpha}$ or $f_b(\,)$ go toward 0, while the distance component goes to infinity. Therefore, it is necessary to ensure that P is not divergent. To do so, we look at the degree of polarization over a more limited sphere of the plane—namely, the diagonal line. Note that this limitation of the surface over which we can examine pairwise polarization does not, in itself, preclude the possibility of divergence. If it were the case that P may go to infinity (because in the integration over the large absolute values of x_a, x_b, y_a, and y_b, the large distance terms cumulatively more than offset the small density terms), then this should be expected for all points along any line in the xy-plane. This would be especially so, because we would not place any restrictions

[8] The interpretation of α as the extent to which the polarization measure departs from the inequality measure (the Gini), as given in ER, is not directly applicable because inequality measures, including the Gini, conventionally refer only to one-dimensional distributions.

on the parameters (variance, correlation coefficient) determining the shape of the distribution in assessing whether the line polarization converges.

First, it will be useful to simplify the multiplicative terms in P because we are considering only points on the primary diagonal—that is, where $y = x$. In addition, without any further loss of generality, let $\mu_x = \mu_y = \mu$. Then the normal distribution, of which the general form is

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot \exp\left[\frac{-1}{2\cdot(1-\rho^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x{}^2} + \frac{(y-\mu_y)^2}{\sigma_y{}^2} - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)\right]$$

can be simplified to

$$f(x, x) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot \exp\left[\frac{-(x-\mu)^2}{2}\left(\frac{\sigma_x{}^2 + \sigma_y{}^2 - 2\rho\sigma_x\sigma_y}{(1-\rho^2)\cdot\sigma_x{}^2\sigma_y{}^2}\right)\right]$$

Define new parameters

$$a = \sqrt{\sigma_x{}^2 + \sigma_y{}^2 - 2\rho\sigma_x\sigma_y}\,,\ b = \sigma_x\sigma_y\sqrt{1-\rho^2}\,,\ \hat{\sigma} = b/a\,,\ c = (\sqrt{2\pi}a)^{-1}$$

Then the function $f(x, x)$ above can be written in a form proportional to a univariate normal density function:

$$f(x, x) = c \cdot \frac{1}{\sqrt{2\pi}\hat{\sigma}} \cdot \exp\left(\frac{-(x-\mu)^2}{2\hat{\sigma}^2}\right) = c \cdot \hat{f}(x)\text{, where } \hat{f}(x) \sim N(\mu, \hat{\sigma}^2)$$

However, this retains the information about the underlying joint distribution through the individual variances of the two variables and the correlation coefficient. Next, simplify the distance term:

$$\| z_a - z_b \| = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2} = \sqrt{(x_a - x_b)^2 + (x_a - x_b)^2} = \sqrt{2}\,| x_a - x_b |$$

Reintroducing these terms into the restricted polarization measure, we get

$$P_{diag} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{2}\,| x_a - x_b |\; f(x_a, x_a)^{1+\alpha}\; f(x_b, x_b)\;\; dx_b\;\, dx_a$$

$$= \sqrt{2}\, c^{2+\alpha} \int_{-\infty}^{\infty} \hat{f}(x_a)^{1+\alpha} \int_{-\infty}^{\infty} | x_a - x_b |\; \hat{f}(x_b)\; dx_b\;\, dx_a$$

Note that the normal distribution raised to a power can be reformulated as a function proportional to a normal function. To see this, let $f(x) \sim N(\mu, \sigma^2)$. Then

$$f(x)^n = \frac{1}{\sqrt{2\pi}^n\sigma^n}\frac{\sqrt{2\pi}\sigma/\sqrt{n}}{\sqrt{2\pi}\sigma/\sqrt{n}}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2/n}\right)$$

$$= q_n \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma_n} \exp\left(\frac{-(x-\mu)^2}{2\sigma_n^{\ 2}} \right) = q_n \cdot f_{(n)}(x)$$

where

$$q_n = \frac{(\sqrt{2\pi}\sigma)^{1-n}}{\sqrt{n}}, \ f_{(n)}(x) \sim N(\mu, \sigma_n^{\ 2}) \text{ and } \sigma_n = \frac{\sigma}{\sqrt{n}}$$

Using this to transform the term raised to (1 + α),

$$P_{diag} = \sqrt{2}\, c^{2+\alpha} q_{1+\alpha} \int_{-\infty}^{\infty} \hat{f}_{(1+\alpha)}(x_a) \int_{-\infty}^{\infty} | x_a \hat{f}(x_b) - x_b \hat{f}(x_b) | \, dx_b \ dx_a$$

With the basic rule that the sum of the absolute value of two variables is greater or equal to the absolute value of the subtraction of one variable from the other, we can establish an upper bound for P_{diag} and show that this upper bound is not divergent:

$$P_{diag} \leq P_{diag}^{+} = \sqrt{2}\, c^{2+\alpha} q_{1+\alpha} \int_{-\infty}^{\infty} \hat{f}_{(1+\alpha)}(x_a) \int_{-\infty}^{\infty} | x_a \hat{f}(x_b) | + | x_b \hat{f}(x_b) | \, dx_b \ dx_a$$

$$= \sqrt{2}\, c^{2+\alpha} q_{1+\alpha} \int_{-\infty}^{\infty} \hat{f}_{(1+\alpha)}(x_a) \cdot \left(| x_a | \int_{-\infty}^{\infty} \hat{f}(x_b)\, dx_b + \int_{-\infty}^{\infty} | x_b | \, \hat{f}(x_b) \ dx_b \right) dx_a$$

$$= \sqrt{2}\, c^{2+\alpha} q_{1+\alpha} \int_{-\infty}^{\infty} \hat{f}_{(1+\alpha)}(x_a) \cdot \left(| x_a | + \int_{-\infty}^{0} (-x_b)\, \hat{f}(x_b)\, dx_b + \int_{0}^{\infty} x_b \hat{f}(x_b) \ dx_b \right) dx_a$$

The expected value can be expressed as a weighted sum of two integrals: Again let $f(x) \sim N(\mu, \sigma^2)$. Then,

$$\mu = \int_{-\infty}^{0} x \, f(x)\, dx + \int_{0}^{\infty} x \, f(x)\, dx = r\mu + (1-r)\mu$$

where

$$r = \int_{-\infty}^{0} f(x)\, dx = \Phi(-\mu / \sigma)$$

with $\Phi($ $)$ denoting the cumulative density function of the standard normal distribution. Expressing the integrals with absolute-value terms as additive integrals over the positive and negative range, respectively, and making use of the above way of writing the expected value, we have

$$P_{diag}^{+} = \sqrt{2}\, c^{2+\alpha} q_{1+\alpha} \int_{-\infty}^{\infty} \hat{f}_{(1+\alpha)}(x_a) \cdot \left(| x_a | - \Phi(-\mu / \hat{\sigma}) \cdot \mu + \Phi(\mu / \hat{\sigma}) \cdot \mu \right) dx_a$$

$$= \sqrt{2}\, c^{2+\alpha} q_{1+\alpha} \int_{-\infty}^{\infty} \hat{f}_{(1+\alpha)}(x_a) \cdot | x_a | \cdot (2 \cdot \Phi(\mu / \hat{\sigma}) - 1) \cdot \mu \ dx_a$$

$$= \sqrt{2}\, c^{2+\alpha} q_{1+\alpha} (2\,\Phi(\mu/\hat{\sigma})-1)\cdot \mu \cdot \left(\int_{-\infty}^{0} (-x_a)\cdot \hat{f}_{(1+\alpha)}(x_a)\, dx_a + \int_{0}^{\infty} x_a \cdot \hat{f}_{(1+\alpha)}(x_a)\, dx_a \right)$$

$$= \sqrt{2}\, c^{2+\alpha} q_{1+\alpha} \mu^2 \cdot (2\cdot\Phi(\mu/\hat{\sigma})-1)\cdot(2\cdot\Phi(\mu/\hat{\sigma}_{1+\alpha})-1)$$

which clearly gives a finite value. Therefore, the polarization measure for all points along the diagonal in the *xy*-plane does not go to infinity as we consider the infinite range of points on the diagonal line.

Invariance of the Measure P

Certain distributional changes generally do not affect the polarization measure P. These include all changes in distribution that retain the shape of the distribution but shift its location or position. The central location of any distribution is described by its mean. In this sense, any change in $[\mu_x, \mu_y]$ will not change P. Neither will a change in position. Positional changes only apply to multivariate distributions. An example of a 'positional change', in the sense used here, is a change in a bivariate distribution that retains both the shape and central location but constitutes a "pivot" around the mean. To formally express a pivot in a specific case, consider the normal distribution that is symmetric about the main diagonal line through its mean point—in other words, the case in which the two variances are equal and the correlation coefficient is nonnegative:

$$f_1(x, y) = \frac{1}{2\pi\sigma_1^2\sqrt{1-\rho_1^2}} \cdot \exp\left(-\frac{(x-\mu_x)^2 + (y-\mu_y)^2 - 2\rho_1(x-\mu_x)(y-\mu_y)}{2\cdot(1-\rho_1^2)\cdot\sigma_1^2} \right)$$

Then, the normal distribution that retains the shape of the above but that is pivoted about its mean takes on the form

$$f_\beta(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot \exp\left[\frac{-1}{2\cdot(1-\rho^2)} \left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \right) \right]$$

Define A, B, and C as follows:

$$A = C + (1-\rho^2)(1-\rho_1)(1+\beta^2) \tag{14a}$$

$$B = A - \sqrt{A^2 - 4\beta^2(1-\rho^2)(1-\rho_1^2)} \tag{14b}$$

$$C = 2\beta\rho\sqrt{1-\rho^2}\sqrt{1-\rho_1^2} \tag{14c}$$

The second *moment* parameters in $f_\beta(x, y)$ are specified as functions of the original distribution's parameters σ_1, ρ_1, and β. The parameter β indicates the extent of the pivot and is, in essence, the slope of the line through the mean about which the distribution is symmetric. Thus, for example, $\beta = 0.7$ constitutes a pivot to the right of the above "diagonal" normal distribution, and $f_\beta(x, y; \beta = 1) = f_1(x, y)$. Specifically, the variances of the pivoted function are

$$\sigma_y = \frac{\sigma_1^2\sqrt{1-\rho_1^2}}{\sigma_x\sqrt{1-\rho^2}} \tag{15}$$

and

$$\sigma_x = \left(\frac{\sigma_1{}^2 B}{2\beta^2 (1-\rho^2)} \right)^{1/2} \tag{16}$$

and ρ is implicitly defined in that it takes on the value that satisfies the equality

$$B = C \tag{17}$$

(Appendix B contains a proof of the validity of $f_p(x, y)$ as a pivot transformation of the corresponding "diagonal" distribution). A pivoted bimodal distribution ½($f(x, y, \mu_x, \mu_y) + f(x, y, \mu_x', \mu_y')$) can be constructed in very analogous fashion and by also appropriately changing the mean points of each component distribution.

It is easy to see why shifts of the central location of a distribution, as well as pivots (and any combination of a shift and pivot), leave P unchanged. For every pair point $[(x_a, y_a), (x_b, y_b)]$ in the original function that generates the social distance $\sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}$ and the densities $f(x_a, y_a)$ and $f(x_b, y_b)$, there exists a corresponding point pair with the same distance $[(x_a', y_a'), (x_b', y_b')]$ in the transformed function, generating the same distance and function values. The only difference in the latter point pair are the values of the x- and y-variables, which are, after accounting for the absolute distance, immaterial to the polarization measure P.

The Polarization Parameter α

The measure in (13) is explicitly denoted as a function only of α, because the variables x_a, y_a, x_b, and y_b disappear in the process of integration. The measure decreases with greater values of α for certain kinds of distributions. These distributions are characterized by relative dispersion of the population in both dimensions, as well as relatively limited correlation between x and y. This can be argued from examining the derivative $\partial P/\partial\alpha$:

$$P'(\alpha) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}\, f(x_a, y_a)^{1+\alpha} \cdot \ln(f(x_a, y_a))\, f(x_b, y_b)\;\; dx_a dx_b dy_a dy_b$$

The term to be integrated is negative if $f(x_a, y_a) < 1$. In the case of a normal distribution, the mode is $f(x{=}\mu_x, y{=}\mu_y) = \left(2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}\right)^{-1}$, so that all values of this distribution function are less than 1 for values of σ_x and σ_y that are not too small and values of ρ that are not too large. Except for extreme values of the variances and correlation coefficient, the polarization measure decreases with larger choices of values for the parameter α.

Perhaps more important than how the size of α affects the size of the measure is the fact that the value chosen for α modifies to some extent the polarization ranking of different distributions, which has implications on how α has to be restricted for P to satisfy certain polarization criteria. This too will be discussed in the next section.

6. THE MEASURE *P* AND THE AXIOMS OF POLARIZATION

In expressing polarization criteria as formal axioms, Section 4 related these axioms to first and second moments of unimodal normal distributions and to the local moments of bimodal distributions. Using axioms facilitates an examination of the extent to which the polarization measure P, defined on the same uni- and bimodal distributions, satisfies the axioms, as will be done in this section.

We examine the value of the polarization measure P in both a normal distribution and a bimodal distribution with normal additive components (see Appendix C for details of the numerical analysis). Figures 5 and 6 show that lower variance distributions are associated with greater values of P, both in the unimodal case and the bimodal case. This holds under some restrictions on α; specifically, as long as $\alpha > 0.52$ in the unimodal and $\alpha > 0.3$ in the bimodal distribution, the numerical results suggest that P always decreases in the local variances for all values of ρ. Therefore, given a suitable lower bound on α in the polarization measure P, the concentration criterion for polarization is satisfied by P.

Inequality (12b)—generalizing (12a) and reversing its sign—identifies those regions in xy-space for which a normal distribution increases when the distribution narrows (i.e., σ falls):

$$(x-\mu_x)^2 + (y-\mu_y)^2 - 2\,\rho\,(x-\mu_x)(y-\mu_y) < 2\,\sigma^2(1-\rho^2) \qquad (12b)$$

These are points within an ellipse centered at $[\mu_x, \mu_y]$. From the right side of the inequality, it is apparent that the increasing region becomes smaller when the distribution further narrows. Compare two distributions with respective variances σ_{lo} and σ_{hi}, where $\sigma_{lo} < \sigma_{hi}$. Since in $f_{lo}(\ )$, the area of increasing density is smaller than in $f_{hi}(\ )$ with a marginal decrease in σ, but also since the overall volume under the function remains constant at 1, the increasing portion of f_{lo} must, on average, be rising at a faster rate than the increasing portion of f_{hi}. In other words, we must have

$$\iint_{[x,y]\in A_{lo}} \frac{\partial^2 f_{lo}}{(\partial\sigma)^2}\cdot f_{lo}\ dx\,dy \;>\; \iint_{[x,y]\in A_{hi}} \frac{\partial^2 f_{lo}}{(\partial\sigma)^2}\cdot f_{lo}\ dx\,dy \qquad (18)$$

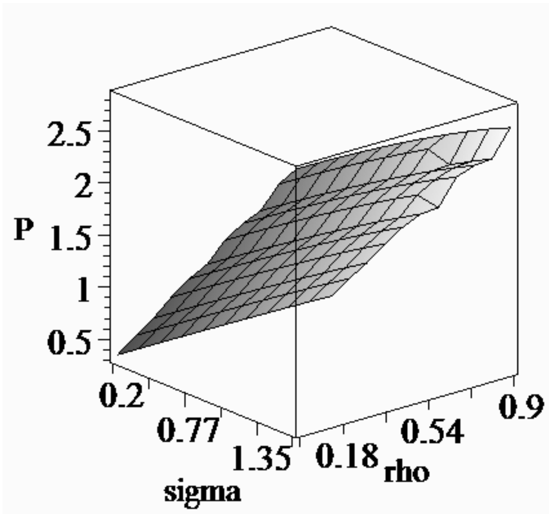
where $A_i = \{x, y \mid (x-\mu_x)^2 + (y-\mu_y)^2 - 2\rho\,(x-\mu_x)(y-\mu_y) < 2\sigma_i^2(1-\rho^2)\}$, $i = \{lo, hi\}$.

At the same time, the increasing areas as the distribution narrows are also the highest density areas for any level of σ. When the polarization parameter α is relatively large, it shifts more weight from low- to high-density areas as compared with the case of a small α. Therefore, for a sufficiently large value of α, the polarization measure P increases when the distribution becomes more concentrated, because the areas of increased density contribute more to the “sum” of all levels of identification than the areas of reduced density take away.

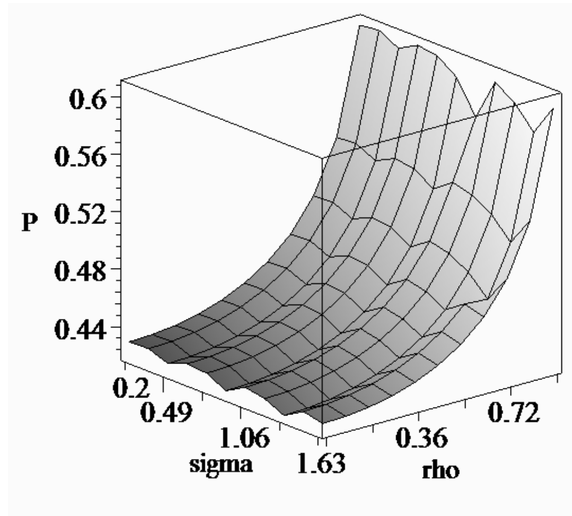
In Figures 5 and 6, we see that there is a positive monotonic relationship between the degree of correlation between the two identity variables, on the one hand, and the polarization measure, on the other. As before, this holds for large enough values of α for both the unimodal and the bimodal distributions, where the threshold level of α is similar to that above which $dP/d\sigma$ is nonincreasing.

Figure 5. Degree of polarization for different values of σ, ρ, and α in a unimodal distribution

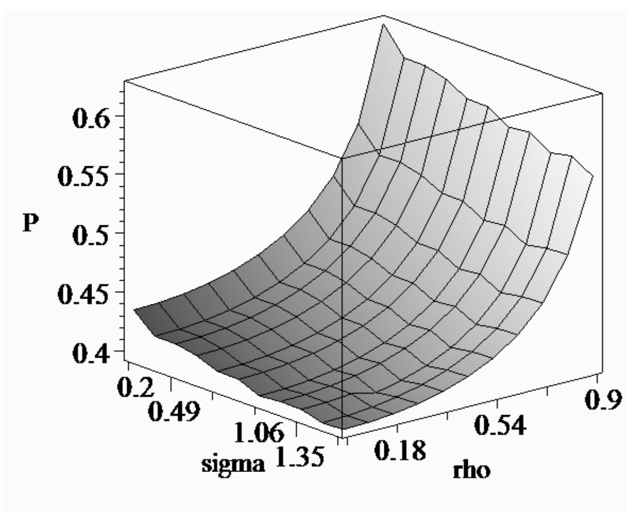
Unimodal distribution, $\alpha = 0$



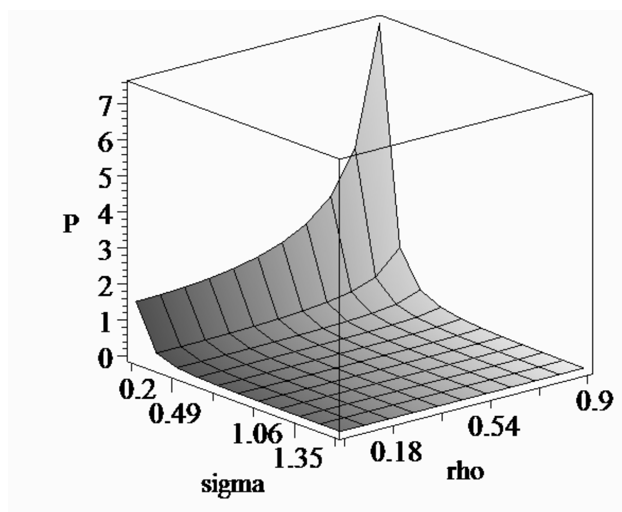
Unimodal distribution, $\alpha = 0.5$



Unimodal distribution, $\alpha = 0.52$

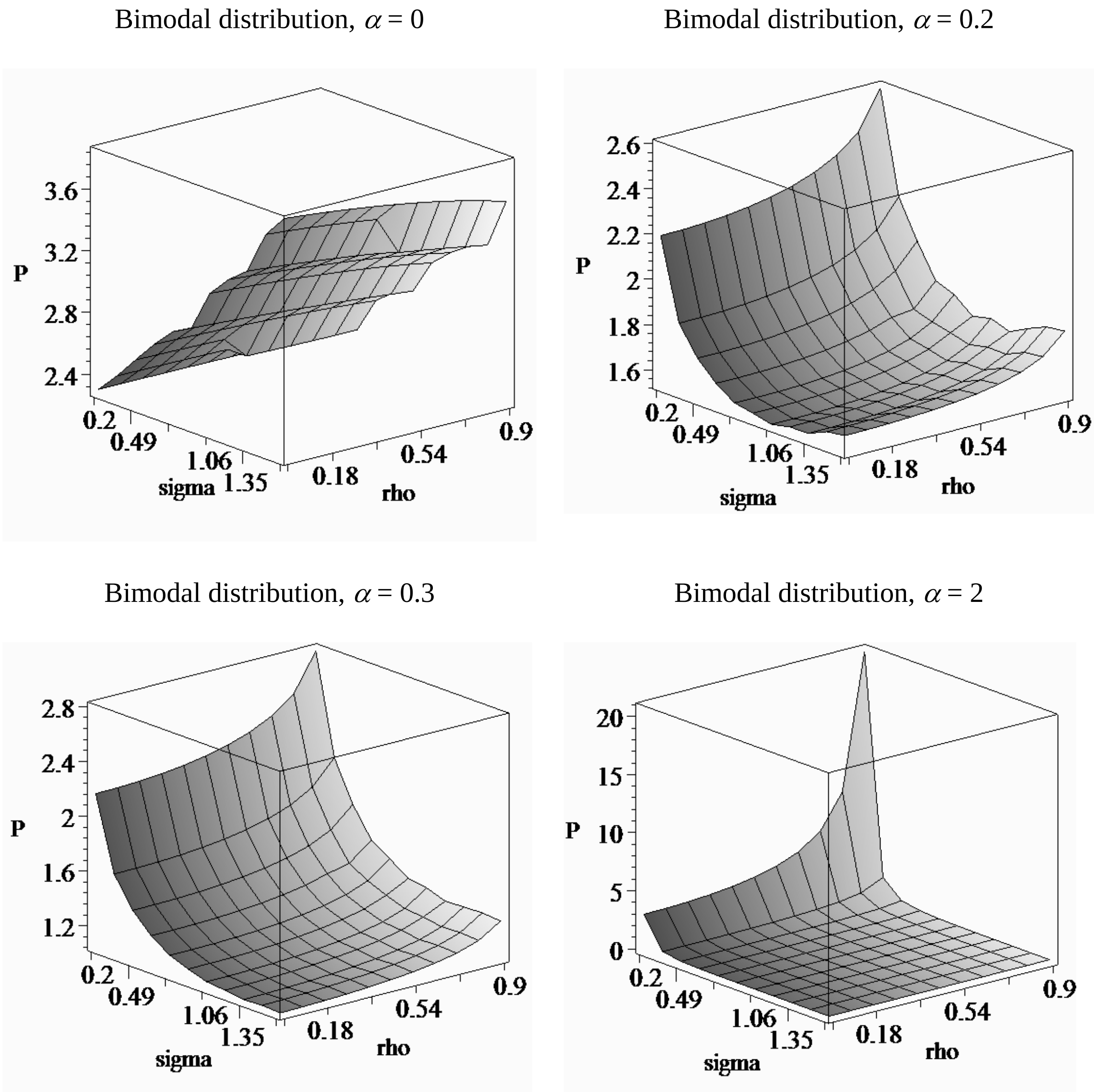


Unimodal distribution, $\alpha = 2$



From the same graphs, we also see that the greater correlation between the variables *x* and *y* increases polarization for the full range of variances considered (although the increase in *P* with an increase in ρ becomes much less pronounced when the distribution is more dispersed). Given this, as in the case of the relationship between the polarization measure and the dispersion of the distribution, with a lower-bound restriction on α, *P* satisfies the correlation criteria of polarization. The logic behind the impact of the correlation between *x* and *y* on *P* is directly related to the reasoning behind the higher *P* in a distribution with less dispersion.

Figure 6. Degree of polarization for different values of σ, ρ, and α in a bimodal distribution



In both cases, the "concentration" of the population leads to a reweighing of each identification in a way that this component contributes more to overall polarization. Indeed, the link between how an increase in correlation and a fall in the variances affect polarization can be made much more immediate. Recall from Section 5 that any bivariate normal distribution $f_1(x, y)$ with equal variances $\sigma_x = \sigma_y = \sigma_1$ and correlation ρ_1 can be "pivoted" so that its major axis has some slope β, retaining its basic shape, by generating another normal distribution $f_\beta(x, y)$ with each parameter σ_x, σ_y, and ρ expressed as a function of the original parameters σ_1 and ρ_1.

To relate the signs of $P'(\rho)$ and $P'(\sigma)$, making use of the pivot framework, let $\rho_0 \in (0, 1)$ and compare

$$f_1(x, y) = f(\sigma_x = \sigma_y = 1, \rho = \rho_0) = \frac{1}{2\pi\sqrt{1-\rho_0^{\,2}}} \cdot \exp\left(\frac{(x-\mu_x)^2 + (y-\mu_y)^2 - 2\rho_0 (x-\mu_x)(y-\mu_y)}{-2\cdot(1-\rho_0^{\,2})} \right)$$

with

$$f_0(x, y) = f(\rho = 0) = \frac{1}{2\pi\sigma_x\sigma_y}\cdot\exp\left(\frac{(x-\mu_x)^2}{-2\sigma_x^{\,2}}+\frac{(y-\mu_y)^2}{-2\sigma_y^{\,2}}\right)$$

For σ_x and σ_y to be such that $f_0(x, y)$ is a pivot transformation of $f_1(x, y)$, a necessary condition is that $f_0(\mu_x, \mu_y) = f_1(\mu_x, \mu_y)$—in other words, the modes must be of equal height. Because $f_0(\mu_x,\mu_y) = (2\ \pi\ \sigma_x\ \sigma_y)^{-1}$ and $f_1(\mu_x, \mu_y) = (2\pi(1-\rho^2)^{½})^{-1}$, this generates the first condition,

$$\sigma_x\sigma_y = (1-\rho_0^{\,2})^{½} \tag{19}$$

Similarly, the value of $f_1(x, y)$ on its major axis at some distance $(2^{½}a)$ from the mean must equal $f_0(x, y)$ at the *point* $(2^{½}a)$ to the right (or left) of the mean, because the major axis here is horizontal. More compactly, this condition is $f_1(\mu_x + a,\ \mu_y + a) = f_0(\mu_x + 2^{½}a,\ \mu_y)$; incorporating (19), it becomes

$$\exp\left(\frac{a^2+a^2-2\rho_0a^2}{-2\cdot(1-\rho_0^{\,2})}\right) = \exp\left(\frac{2a^2}{-2\sigma_x^{\,2}}\right)$$

or

$$\sigma_x = \sqrt{1+\rho_0}$$

Together with (19), this gives

$$\sigma_y = \sqrt{1-\rho_0}$$

So, because $f_0 \equiv f(\sigma_x = (1+\rho_0)^{½},\ \sigma_y = (1-\rho_0)^{½},\ \rho = 0)$ is a pivot transformation of $f_1 \equiv f(\sigma_x = \sigma_y = 1,\ \rho = \rho_0)$, the degree of polarization of both distributions is identical—that is,

$$P(f_0) = P(f_1) \tag{20}$$

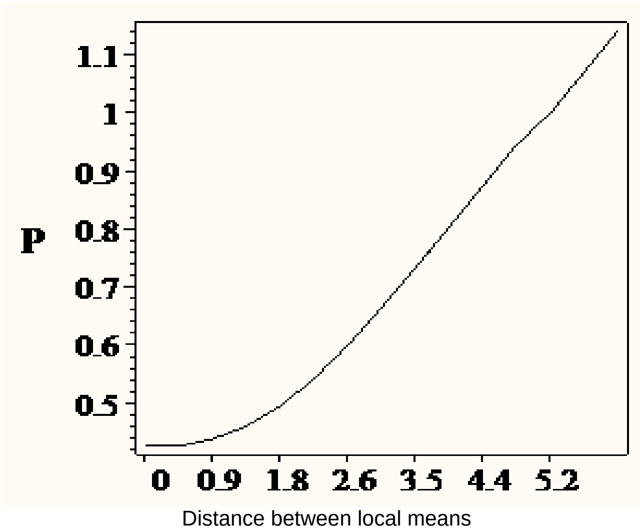
Furthermore, defining $f_{10} \equiv f(\sigma_x = \sigma_y = 1,\ \rho = 0)$, $P'(\rho) > 0$ implies that

$$P(f_1) > P(f_{10}) \tag{21}$$

But noticing that $(1+\rho_0)^{½}(1-\rho_0)^{½} = (1-\rho_0^{\,2})^{½} < 1$, it is also the case that $P'(\sigma^2) < 0$ implies that $P(f_0) > P(f_{10})$, which of course is already clear from (20) and (21).

Finally, Figure 7 presents the relationship between P and the distance between the two poles, $||\mu_{z1} - \mu_{z2}||$, holding other attributes of the distribution constant. As is suggested by the intuition behind the pole separation criterion for polarization, the graph shows increasing polarization with a greater separation between the two components of the distribution. Indeed, the relationship is nearly linear, and P at a distance between the local means of 5.7 is approximately three times the value of P when there is no distance—that is, for the special case of a unimodal normal distribution.

Figure 7. Polarization for different values of local mean separation



P can be unpacked to understand why a mean separation increases polarization. Consider the bimodal distribution with conditions as in (8); in addition, let $\sigma = 1$ and $\rho = 0$. Examining what happens with the degree of alienation between a pair of points when the distance between the local mean increases will shed some light on the impact on overall polarization. We consider the alienation of an individual at some point $a = [x_a, y_a]$ to another at $b = [x_b, y_b]$ in the neighborhood of local means μ_a and μ_b, respectively. Recall that the bivariate analogue to (5), (4), and (8) gives the effective alienation of *a* to *b* as

$$T(z_a, z_b) = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2} \cdot g(x_a, y_a)^\alpha$$
$$= \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2} \cdot \left[\frac{1}{2}\left(f^a(x_a, y_a) + f^b(x_a, y_a)\right)\right]^\alpha$$

where $f^a(\ )$ is the normal density function and the point *a* is in the vicinity of its mean and the analogous holds for $f^b(\ )$ and *b*. The alienation function shows that there are two counteracting forces when the poles of the bimodal distribution drift apart. First, the social distance component increases. Second, the identification component $g(x_a, y_a)^\alpha$ falls, because as the two functions $f^a(\ )$ and $f^b(\ )$ separate, the mass $f^b(x_a, y_a)$ falls (though not $f^a(x_a, y_a)$, because the point *a* moves along with the function μ_a so that the distance between *a* and μ_a remains the same). More concretely, let *k* denote the outward shift along both the *x* and the *y* dimensions. Because from (8), we have $\mu_a = -\mu_b$, we can write the means in terms of μ_b and drop the *b* subscript. Because the outward shift also shifts the points *a* and *b* apart, we have the alienation of *a* toward *b* after the shift as

$$T^{(k)}(z_a, z_b) = \sqrt{((x_a - k) - (x_b + k))^2 + ((y_a - k) - (y_b + k))^2} \cdot$$
$$\cdot \left[\frac{1}{2}\left[f^a((x_a - k), (y_a - k); (-\mu - k)) + f^b((x_a - k), (y_a - k); (\mu + k))\right]\right]^\alpha$$

$$= \sqrt{(x_a - x_b - 2k)^2 + (y_a - y_b - 2k)^2} \cdot \left(\frac{1}{4\pi}\right)^{\alpha} \cdot$$

$$\cdot \left[\exp\left[-0.5((x_a + \mu)^2 + (y_a + \mu)^2)\right] + \exp\left[-0.5((x_a - \mu - 2k)^2 + (y_a - \mu - 2k)^2)\right]\right]^{\alpha}$$

Social distance increases, and the density $f^b(\ )$ decreases, because given that a and b are in the vicinity of $-\mu$ and $\mu > 0$, respectively, we know $x_a < 0 < x_b$. As long as μ (and with that the initial distance between the local means) is sufficiently large, the increase in social distance with greater mean separation more than offsets the density decrease, as Table 1 shows numerically for different points of a.

Table 1. Impact of mean separation on the components of alienation

For $x_b = y_b = \mu = 1$

		$\alpha = 1$			$\alpha = 3$		
		$k = 0$	$k = 0.5$	$k = 1$	$k = 0$	$k = 0.5$	$k = 1$
x_a	y_a	Social distance					
–1.5	–1.5	3.54	4.95	6.36	3.54	4.95	6.36
–1.0	–1.5	3.20	4.61	6.02	3.20	4.61	6.02
–0.5	–1.5	2.92	4.30	5.70	2.92	4.30	5.70
–1.0	–1.0	2.83	4.24	5.66	2.83	4.24	5.66
–0.5	–1.0	2.50	3.91	5.32	2.50	3.91	5.32
–0.5	–0.5	2.12	3.54	4.95	2.12	3.54	4.95
x_a	y_a	Identification					
–1.5	–1.5	0.06213	0.06198	0.06197	2.3981E-04	2.3804E-04	2.3804E-04
–1.0	–1.5	0.07070	0.07023	0.07023	3.5339E-04	3.4637E-04	3.4635E-04
–0.5	–1.5	0.06311	0.06198	0.06198	2.5136E-04	2.3813E-04	2.3804E-04
–1.0	–1.0	0.08103	0.07959	0.07958	5.3213E-04	5.0412E-04	5.0393E-04
–0.5	–1.0	0.07372	0.07027	0.07023	4.0069E-04	3.4692E-04	3.4635E-04
–0.5	–0.5	0.07036	0.06213	0.06198	3.4835E-04	2.3981E-04	2.3804E-04
x_a	y_a	Alienation					
–1.5	–1.5	0.22	0.31	0.39	0.85E-03	1.18E-03	1.51E-03
–1.0	–1.5	0.23	0.32	0.42	1.13E-03	1.60E-03	2.09E-03
–0.5	–1.5	0.18	0.27	0.35	0.73E-03	1.02E-03	1.36E-03
–1.0	–1.0	0.23	0.34	0.45	1.51E-03	2.14E-03	2.85E-03
–0.5	–1.0	0.18	0.27	0.37	1.00E-03	1.35E-03	1.84E-03
–0.5	–0.5	0.15	0.22	0.31	0.74E-03	0.85E-03	1.18E-03

As *P* increases with greater mean distance and lower variance, it also widely satisfies the "shrinking middle" criterion of polarization. Section 4b provided some general conditions under which lower variance shrinks the middle spectrum of a bimodal distribution. These conditions are the initial degree of local variance and the specification of the limits of the middle spectrum. Thus, *P* is in accordance with this criterion for relatively low values of *c*, or more narrowly defined middle spectra.

7. RELATING THE TWO-DIMENSIONAL TO THE ONE-DIMENSIONAL POLARIZATION MEASURE

The introduction provided a motivation for the enterprise of this paper—namely, to develop and describe an explicitly two-dimensional measure of polarization. The young literature on this subject has nearly exclusively preoccupied itself with measuring the degree of polarization of some economic variable, usually income. Given both the motivation of a two-dimensional measure and the usefulness of linking the latter back to the existing literature, the effort of a comparison between P and its univariate counterpart offers itself and is the subject of this section.

Recall the one-dimensional analogue to P from (6):

$$P^y = \int\int | y_a - y_b | \, f(y_a)^{1+\alpha} \, f(y_b) \, dy_a \, dy_b$$

Rewriting $|y_a - y_b|$ as $\sqrt{(y_a - y_b)^2}$ and doing the same for P^x renders the marginal polarizations in a form that facilitates a comparison between the multiplication of the two one-dimensional measures, $P^x \cdot P^y$ and P. Because $P^x \cdot P^y$ naturally cannot capture any correlation between x and y, this comparison is only useful for the case of independence between the two variables.

$$P^x \cdot P^y = \int\int\int\int \sqrt{(x_a - x_b)^2 (y_a - y_b)^2} \left(f(x_a) f(y_a) \right)^{1+\alpha} f(x_b) f(y_b) \; dx_a \, dx_b \, dy_a \, dy_b$$

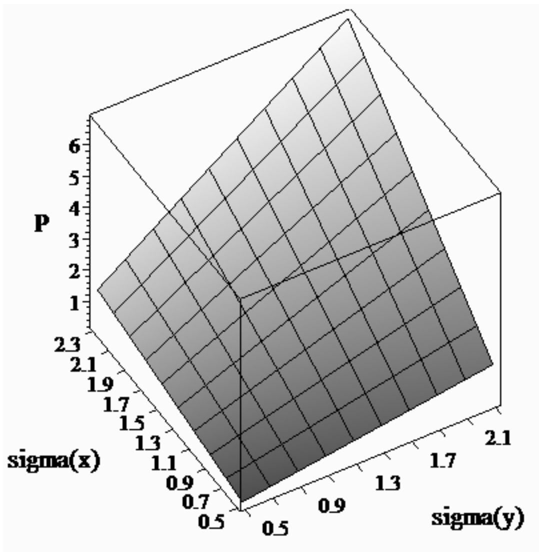
$$P(\rho = 0) = \int\int\int\int \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2} \left(f(x_a) f(y_a) \right)^{1+\alpha} f(x_b) f(y_b) \; dx_a \, dx_b \, dy_a \, dy_b$$

We see that the two measures differ only in the component that measures social distance. For any $w_1, w_2 > 0$, $w_1 + w_2$ tends to be larger than $w_1 \cdot w_2$ whenever either w_1 or w_2 is relatively small in value and $w_1 + w_2 > w_1 \cdot w_2 \; \forall \; w_1, w_2 < 2$. This suggests that the integrated measure P would be larger than (or falls less short of) the multiplicative measure $P^x \cdot P^y$ for distributions with relatively low dispersion and greater concentration. This is because, in such distributions, two points that are relatively far from each other in both their x and y dimensions—that is, where $|x_a - x_b|$ and $|y_a - y_b|$ are large—will be more likely to have a low density. The reduced contribution to overall polarization of the pairwise alienation between these two points when dispersion is lower in the population impacts $P^x \cdot P^y$ more greatly than it does P.

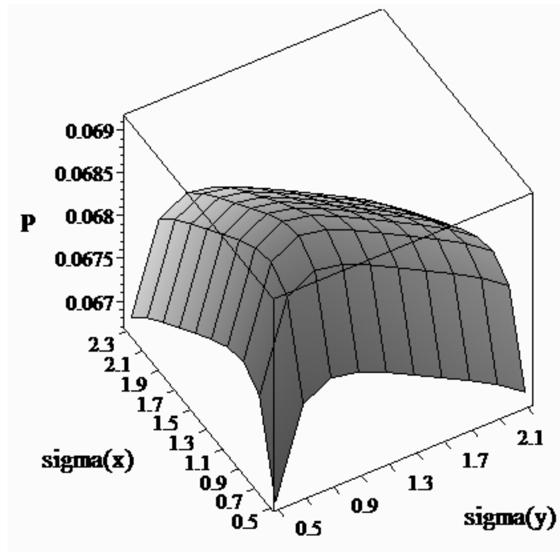
The numerical analysis seems to bear out this intuition. Figure 8 shows the levels of polarization for a range of values of σ_x, σ_y, and α, while Figure 9 displays for which parameter values $P^x \cdot P^y > P$ (vertical value = 1) and vice versa (vertical value = 2).

Figure 8a. Multiplication of the marginal polarizations $P^{1x} \cdot P^{1y}$

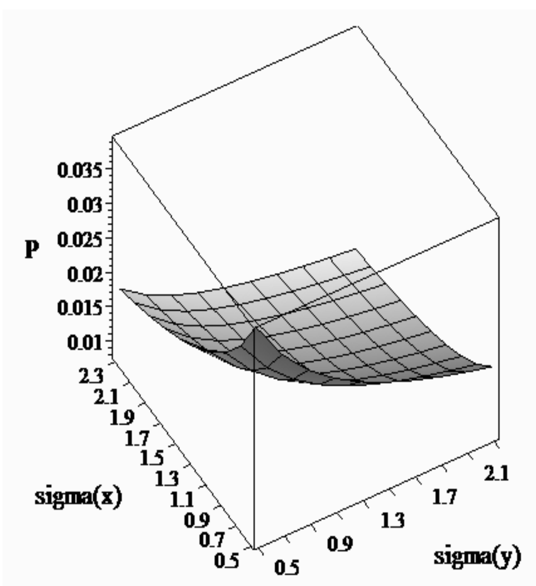
$P^{1x} \cdot P^{1y}$, $\alpha = 0$



$P^{1x} \cdot P^{1y}$, $\alpha = 0.5$



$P^{1x} \cdot P^{1y}$, $\alpha = 1.5$



$P^{1x} \cdot P^{1y}$, $\alpha = 2$

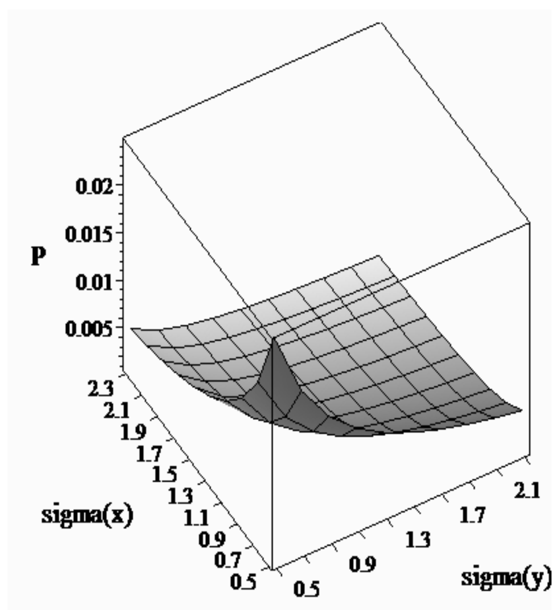
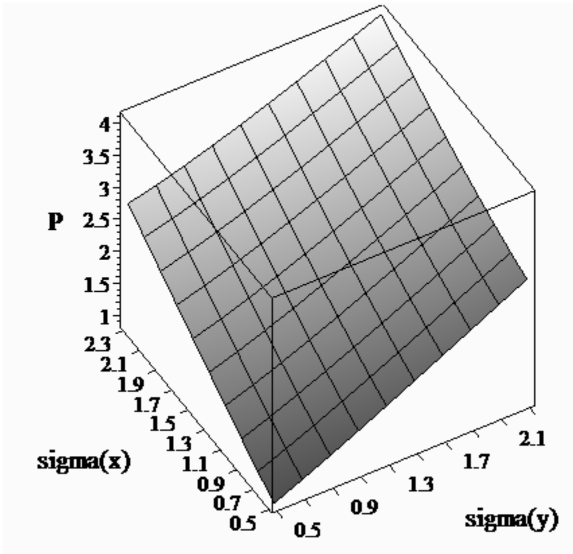
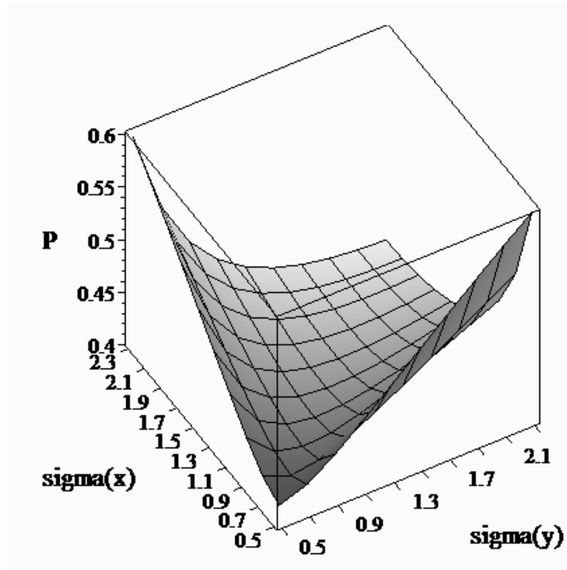


Figure 8b. *P* for various values of σ_x and σ_y

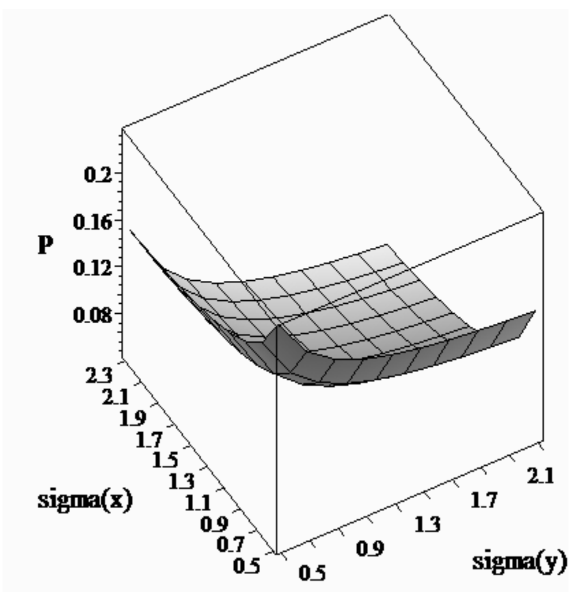
***P* for bivariate dist, $\rho = 0$, $\alpha = 0$**



***P* for bivariate dist, $\rho = 0$, $\alpha = 0.5$**



***P* for bivariate dist, $\rho = 0$, $\alpha = 1$**



***P* for bivariate dist, $\rho = 0$, $\alpha = 2$**

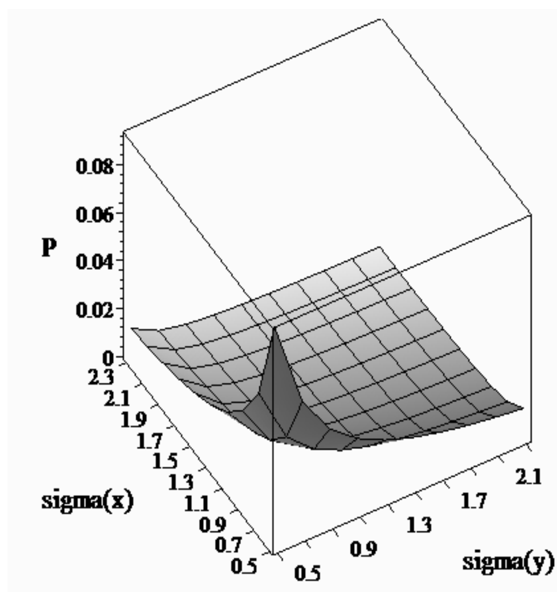
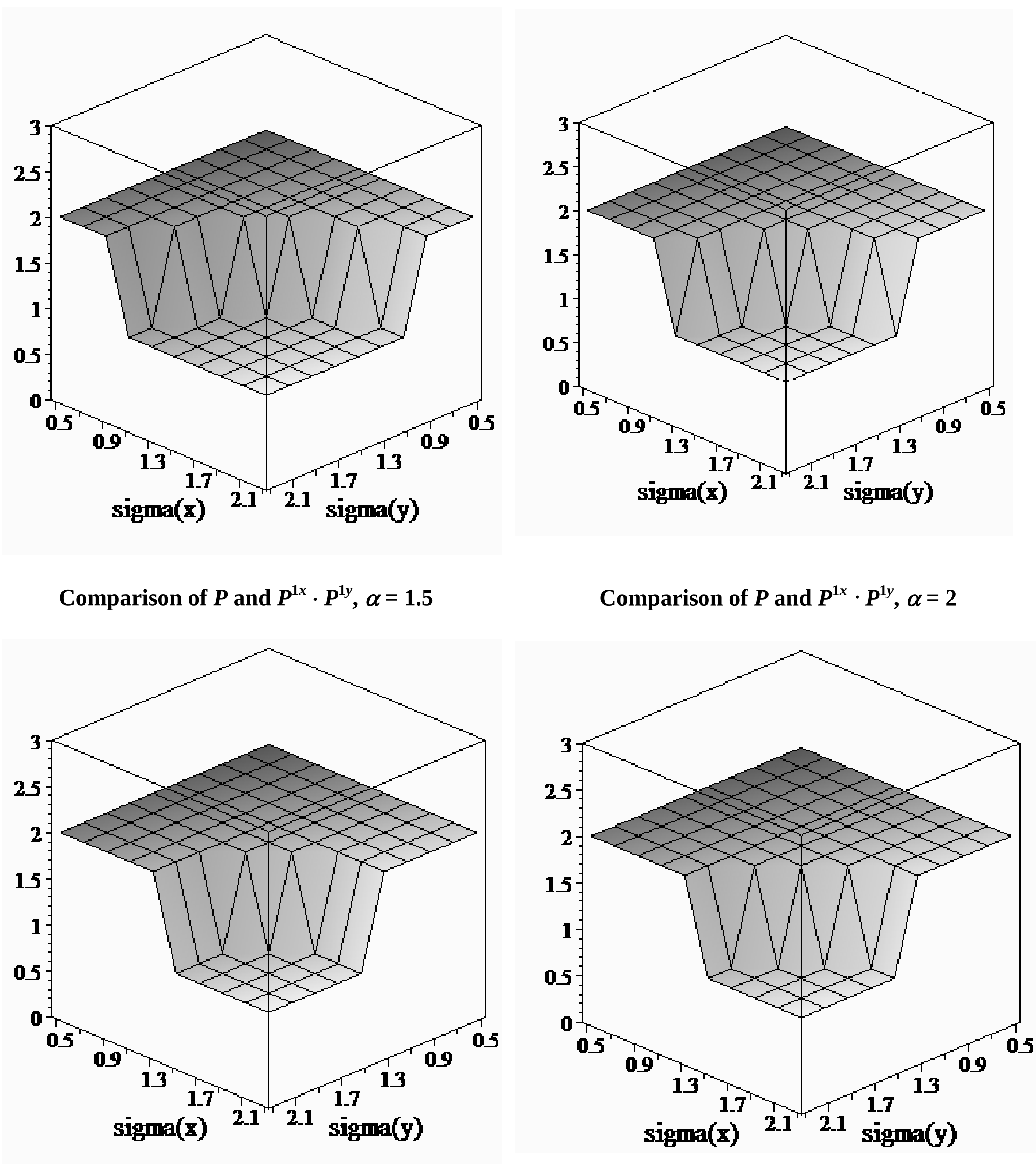


Figure 9. Comparison of polarization levels for P vs. $P^{1x} \cdot P^{1y}$ for a unimodal normal distribution

Comparison of P and $P^{1x} \cdot P^{1y}$, $\alpha = 0$ **Comparison of P and $P^{1x} \cdot P^{1y}$, $\alpha = 1$**

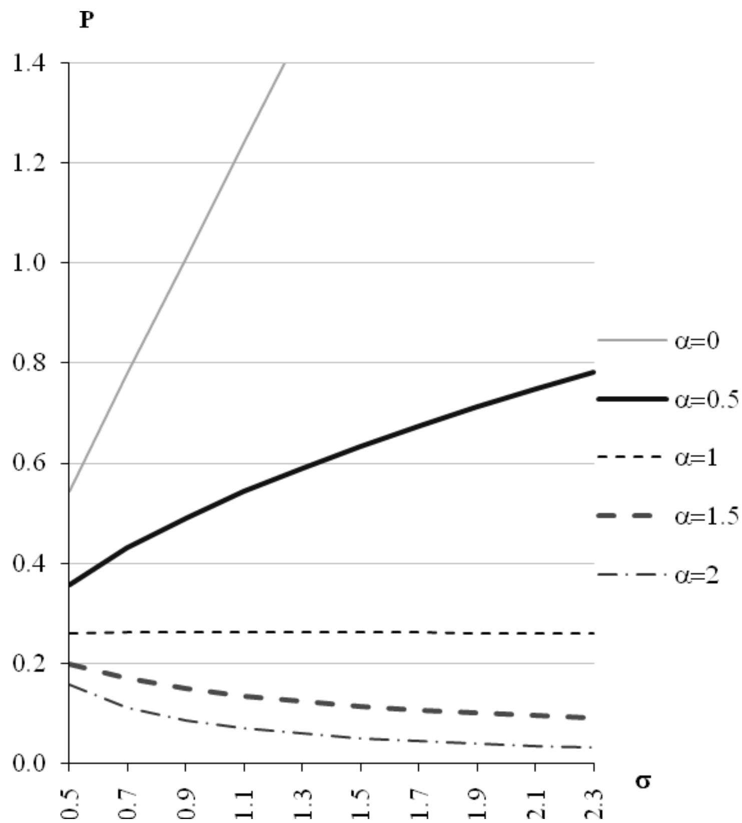
Comparison of P and $P^{1x} \cdot P^{1y}$, $\alpha = 1.5$ **Comparison of P and $P^{1x} \cdot P^{1y}$, $\alpha = 2$**



The figures show that the integrated value P is larger for distributions more concentrated in either the x or the y dimension or both. In light of the criteria elaborated in Section 4—namely, that a measure should reveal more concentrated societies as being more polarized—P appears to pronounce the level of polarization more greatly and, in this sense, comes closer to the notion behind the concentration criterion of polarization.

For distributional changes that permit a comparison between concomitant changes in the one- and the two-dimensional polarization measures, there is no change in the ranking of measures. For example, as Figure 10 shows, just like P, P^y decreases with the degree of dispersion (σ_y) for large enough values of α (approximately for $\alpha > 1$), as well as with α itself. In this sense, the value of P above and beyond what P^x and P^y may tell us lies not so much in qualitatively different propositions about the relative degrees of polarization of different societies (or a society over time), but rather in that it describes the distributional implications of a richer scope of societal changes in a way that may be important, especially for exploring the potential consequences of conflict of social polarization and its economic repercussions.

Figure 10. P^y for a unimodal normal distribution



8. CONCLUDING REMARKS

This paper proposes a measure of the degree of polarization of a population distribution over two dimensions, or variables. This extension of the univariate measures of polarization proposed in Esteban and Ray (1994) and Duclos et al. (2004) is motivated by the suggestion that various forms of conflict which can arise from societies that are highly polarized economically are even more likely to emerge when economic distribution intersects with social characteristics. Although the two dimensions proposed here can be applied to such a case—for example, with one dimension representing wealth and the other racial affiliation—the polarization measure is expressed in general terms and thus may be applicable to the distribution over any two variables.

The measure is discussed in light of four axioms that specify the types of distributional changes that should reasonably translate into a higher degree of polarization. Applying the measure to a family of functions that can represent both unimodal and bimodal population distributions, the measure satisfies the four axioms—briefly summarized as a shrinking of the middle class, greater concentration of the population around poles, greater distance between the poles, and higher correlation between the two variables—under certain parametric restrictions.

The last criterion relates specifically to a context in which changes in distribution over multiple variables are observed—that is, a context that cannot be captured by univariate measures of polarization. For the case of a distribution characterized by independence between the two variables, so that the joint distribution can be characterized fully in terms of the two marginal distributions, we compare the polarization measure P with an alternative measure that is a function of the two one-dimensional analogues to P. In such a case, given parametric restrictions, there are no qualitative differences between the two measures, in that polarization rankings do not change.

In introducing a social dimension to distribution, a natural question that arises is whether a continuous metric is appropriate to represent this dimension. The answer certainly depends on the social characteristic of interest. Variables such as “skin color” in racially diverse societies where color helps determine identity and socioeconomic status may reasonably be modeled using a continuous variable. Even in contexts in which the social attribute of interest appears discrete on the surface, a continuous treatment in the model may be feasible, and sometimes even more apt. For example, individuals in a society with two dominant religions could be modeled as being located along a scale from fundamentalist in one religion to fundamentalist in the other religion, with degrees of moderate belief in the two religions, as well as secular position, lying in between. However, there will still be applications in which a discrete variable may most appropriately capture the context, which suggests the development of different models, with this paper forwarding one such model appropriate for some, but certainly not all, situations.

The choice of continuity compels an ordinal interpretation of the social variable. For the examples cited, as well as for many others, this is, in fact, quite natural and intuitive. In light of the underlying motivation of this paper’s undertaking, it is essential, as we seek to capture “social distance” (or, more fittingly, the socioeconomic distance) not merely in terms of difference in incomes between clusters of individuals but also the distance in both the social and the economic spheres. In taking this approach, this paper offers a contribution to the polarization literature, which has captured distance only in economic terms. (This applies both to the more established univariate-measure literature and to the few papers that have forwarded a multidimensional measure of polarization.)

However, building on this model, the social distance measure can be further refined. For example, the absolute scaling discussed in this paper assumes that the distance is invariant to the overall mean value. Therefore, depending on the specific dimensions being investigated, absolute values may have to be transformed to log values, which is a relatively straightforward adjustment. More fundamentally, however, for such a polarization measure to have empirical meaning in some given context, a more sophisticated way of accounting for social difference is called for. Put succinctly, distance does not always map neatly into difference. This is true not only for standard economic variables but also for noneconomic attributes, especially those that are difficult to rank order.

APPENDIX A: CONDITION FOR BIMODALITY OF AN ADDITIVE NORMAL DISTRIBUTION

True bimodality requires that the value of the function at the local mean point be larger than the function's value at the global mean point. In the additive function in (9), the density at the global mean point (0, 0) is

$$f_{Bglo} = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \cdot \exp\left(\frac{-1}{1+\rho}\left(\frac{\mu}{\sigma}\right)^2\right)$$

and the density at each of the local means (μ, μ) and ($-\mu$, $-\mu$) is

$$f_{Bloc} = \frac{1}{4\pi\sigma^2\sqrt{1-\rho^2}} \cdot \left[\exp\left(\frac{-4}{1+\rho}\left(\frac{\mu}{\sigma}\right)^2\right)+1\right]$$

Bimodality then implies that

$$\frac{1}{4\pi\sigma^2\sqrt{1-\rho^2}} \cdot \left[\exp\left(\frac{-4}{1+\rho}\left(\frac{\mu}{\sigma}\right)^2\right)+1\right] > \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \cdot \exp\left(\frac{-1}{1+\rho}\left(\frac{\mu}{\sigma}\right)^2\right)$$

which can be simplified to

$$z^{-3} + z - 2 > 0, \text{ where } z \equiv \exp\left(\frac{1}{1+\rho}\left(\frac{\mu}{\sigma}\right)^2\right) \tag{A1}$$

Solving $z^{-3} + z - 2 = 0$ for z gives

$$z_0 = \left\{1, \frac{1}{3}\left(w_2 + \frac{4}{w_2} + 1\right)\right\}, \text{ where } w_2 = \left(19 + 3\sqrt{33}\right)^{1/3}$$

(A1) implies that the solution must satisfy $z < 1$ or $z > \frac{1}{3}\left(w_2 + \frac{4}{w_2} + 1\right)$. The former is not possible, because the expression in the exponential must be nonnegative. Therefore, the condition for bimodality is limited to the latter inequality.

APPENDIX B: A "PIVOT" TRANSFORMATION OF THE BIVARIATE NORMAL DISTRIBUTION

Consider the normal density function with mean at the point of origin, some correlation ρ_1, and symmetric about the diagonal, with the latter being the case when $\sigma_x = \sigma_y = \sigma_1$. For the points along the main diagonal—for points where $y = x$—the function becomes

$$f_1(x, x) = \frac{1}{2\pi\sigma_1^{\,2}\sqrt{1-\rho_1^{\,2}}} \cdot \exp\left(\frac{-x^2}{2}\frac{\sigma_1^{\,2}+\sigma_1^{\,2}-2\rho_1\sigma_1^{\,2}}{(1-\rho_1^{\,2})\cdot\sigma_1^{\,2}\sigma_1^{\,2}}\right)$$

$$= \frac{1}{2\pi\sigma_1^{\,2}\sqrt{1-\rho_1^{\,2}}} \cdot \exp\left(-\frac{x^2}{(1+\rho_1)\cdot\sigma_1^{\,2}}\right) \tag{A2}$$

We want to construct a distribution that has the same shape as a normal distribution with variances σ_1^2 and correlation ρ_1. Because the central location of the transformed function does not play a role in this analysis, we can set it equal to that of the original function so that the mean of the pivoted function is also the point of origin. Then, the transformed function is to be symmetric about a new line. Let this line be defined by $y = \beta x$. Given that the two distributions share the same shape, the mass at any point on the diagonal line $y = x$ will have to be equal to the mass of the transformed function at a point on the line $y = \beta x$ that has the same distance from the mean. In the diagonal function, consider some point on the diagonal line, $[x, x]$. The distance d between this point and the mean [0, 0] is

$$d = \sqrt{x^2 + x^2} = \sqrt{2}x$$

The point $[x_\beta, \beta x_\beta]$ on the symmetry line of the transformed function that has the same distance to the mean is determined by $d = \sqrt{x_\beta^{\,2} + (\beta x_\beta)^2}$, so $x_\beta = \sqrt{\frac{2}{1+\beta^2}}x$ and $y_\beta = \sqrt{\frac{2}{1+\beta^2}}\beta x$. Equality of the mass at equivalent points between the original and transformed density function means that

$$f(x, x, 0, 0, \sigma_1, \sigma_1, \rho_1) = f\left(\sqrt{\frac{2}{1+\beta^2}}x, \sqrt{\frac{2}{1+\beta^2}}\beta x, 0, 0, \sigma_x, \sigma_y, \rho\right) \tag{A3}$$

The right side is

$$f_\beta(\cdot) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot \exp\left(\frac{-x^2}{(1-\rho^2)(1+\beta^2)}\left(\frac{1}{\sigma_x^{\,2}} + \frac{\beta^2}{\sigma_y^{\,2}} - 2\rho\frac{\beta}{\sigma_x\sigma_y}\right)\right) \tag{A4}$$

Then, for (A3) to hold for any value x, both the denominator of the normal density functions and the numerator (the exponential expression) must be equal. For the former,

$$\frac{1}{2\pi\sigma_1^{\,2}\sqrt{1-\rho_1^{\,2}}} = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

requires that

$$\sigma_y = \frac{\sigma_1^{\ 2}\sqrt{1-\rho_1^{\ 2}}}{\sigma_x\sqrt{1-\rho^2}} \tag{A5}$$

To determine σ_x, equate the exponential terms of (A2) and (A4) to solve for σ_x:

$$\exp\left(-\frac{x^2}{(1+\rho_1)\cdot\sigma_1^{\ 2}}\right) = \exp\left(\frac{-x^2}{(1-\rho^2)(1+\beta^2)}\left(\frac{1}{\sigma_x^{\ 2}}+\frac{\beta^2\sigma_x^{\ 2}\sqrt{1-\rho^2}}{\sigma_1^{\ 4}(1-\rho_1^{\ 2})}-\frac{2\rho\beta\sqrt{1-\rho^2}}{\sigma_1^{\ 2}\sqrt{1-\rho_1^{\ 2}}}\right)\right)$$

or

$$1 = \frac{(1+\rho_1)\sigma_1^{\ 2}\left(\sigma_1^{\ 4}(1-\rho_1^{\ 2})+\beta^2\sigma_x^{\ 4}(1-\rho^2)-2\rho\sigma_1^{\ 2}\beta\sigma_x^{\ 2}\sqrt{1-\rho^2}\sqrt{1-\rho_1^{\ 2}}\right)}{(1-\rho^2)(1-\rho_1^{\ 2})(1+\beta^2)\sigma_x^{\ 2}\sigma_1^{\ 4}}$$

Expressing this as a polynomial in σ_x^2,

$$0 = \sigma_1^{\ 4}(1-\rho_1^{\ 2}) - \sigma_1^{\ 2}\left(2\rho\beta\sqrt{1-\rho^2}\sqrt{1-\rho_1^{\ 2}}+(1-\rho^2)(1-\rho_1)(1+\beta^2)\right)\sigma_x^{\ 2}+\beta^2(1-\rho^2)\sigma_x^{\ 4}$$

Employing the quadratic formula gives the variance for x:

$$\sigma_x^{\ 2} = \frac{\sigma_1^{\ 2}\left(A\pm\sqrt{A^2-4\beta^2(1-\rho^2)(1-\rho_1^{\ 2})}\right)}{2\beta^2(1-\rho^2)} \tag{A6}$$

with

$$A = 2\beta\rho\sqrt{1-\rho^2}\sqrt{1-\rho_1^{\ 2}}+(1-\rho^2)(1-\rho_1)(1+\beta^2) \tag{A7}$$

We see that (A7) is a function of the diagonal distribution's parameters σ_1, ρ_1, the pivot parameter β, and the pivot function's correlation coefficient ρ.

Apart from equality of the density between the original and transformed functions along their respective symmetry lines, a further condition must be fulfilled: For $\rho_1 = 0$ (and therefore $\rho = 0$), any transformed function must be symmetric about any line through the mean, including the main diagonal $y = x$. In other words, for $\rho = 0$, we must have $\sigma_x = \sigma_x = \sigma_1$ for any pivot β. Only one of the two solutions in (A6) satisfies this. To see this,

$$\sigma_x^2(\rho_1 = \rho = 0) = \frac{\sigma_1^{\ 2}}{2\beta^2}\left((1+\beta^2)\pm\sqrt{(1+\beta^2)^2-4\beta^2}\right)$$

$$= \frac{\sigma_1^{\ 2}}{2\beta^2}\left((1+\beta^2)\pm(1-\beta^2)\right) = \begin{cases}\left(\sigma_1^{\ 2}/(2\beta^2)\right)\cdot 1\\ \left(\sigma_1^{\ 2}/(2\beta^2)\right)\cdot(2\beta^2)\end{cases} = \begin{cases}\sigma_1^{\ 2}/(2\beta^2)\\ \sigma_1^{\ 2}\end{cases}$$

Therefore, the unique solution is

$$\sigma_x^{\,2} = \frac{\sigma_1^{\,2}\left(A - \sqrt{A^2 - 4\beta^2(1-\rho^2)(1-\rho_1^{\,2})}\right)}{2\beta^2(1-\rho^2)}$$

It still remains to determine the correlation parameter of the pivoted function. This is done in the course of ensuring that the function satisfies a further condition. So far, the pivoted function has the same density along its pivot axis $y = \beta x$ as the diagonal function along $y = x$; however, this is only indeed a true pivot transformation retaining the shape of the diagonal normal distribution if the new distribution is symmetric about the pivot line. Symmetry about this line is equivalent to the function taking on the highest value at $y = \beta x$ for any x. Let $y^*(x, \cdot)$ be this maximum density; it is found by maximizing the density function with respect to y:

$$y^*(x, \cdot) = \operatorname{argmax}\ f_p(x, y, 0, 0, \sigma_x, \sigma_y, \rho) = \operatorname{argmax}\left(\frac{y^2}{\sigma_y^{\,2}} - \frac{2\rho}{\sigma_x\sigma_y}xy\right) = \frac{\sigma_y}{\sigma_x}\rho\, x$$

Therefore, the pivot slope β must be equal to the slope of the value-maximizing line, $\frac{\sigma_y}{\sigma_x}\rho$:

$$\beta = \frac{\sigma_y}{\sigma_x}\rho$$

Using expressions in (A6), (A5), and (A7),

$$2\rho\beta\sqrt{1-\rho_1^{\,2}}\sqrt{1-\rho^2} = A - \sqrt{A^2 - 4\beta^2(1-\rho^2)(1-\rho_1^{\,2})}$$

Although one cannot explicitly solve for ρ, the above equation gives the implicit solution $\rho(\beta, \sigma_1, \rho_1)$. Using this in (A6) and (A5) yields all parameters as a function of β, σ_1 and ρ_1.

The parameters of the pivot function—σ_x, σ_y, ρ, β—corresponding to the shape parameters of the original diagonal function—σ_1 and ρ_1—have been determined based on distributions centered at the point of origin, but naturally apply to normal distributions with any central location because, as discussed in Section 5, shifts in the distribution do not affect the shape of a distribution.

APPENDIX C: PARAMETER VALUES IN NUMERICAL ANALYSIS

For analysis of how *P* changes with the variance (or local variances) and the degree of correlation between *x* and *y* of the distribution: In the unimodal case, the global mean is $[\mu_x, \mu_y] = [0, 0]$, and variances of *x* and *y* are held equal; therefore, $\sigma = \sigma_x = \sigma_y$. In the bimodal case, the local means are $[\mu_{x1}, \mu_{y1}]$ = [1.5, 1,5] and $[\mu_{x2}, \mu_{y2}]$ = [–1.5, –1,5]; so the global mean is $\mu_g = 0$, and the distance between the local means is $\| \mu_{z1} - \mu_{z2} \| = \sqrt{2} \cdot 3$. Here, too, $\sigma = \sigma_{x1} = \sigma_{y1} = \sigma_{x2} = \sigma_{y2}$. The numerical analysis involves the ranges for the distribution parameters σ and ρ and for the polarization parameter α, as given in the first three columns of Table C.1. For assessing how *P* changes with the distance between local means, the range of distances between global means is given in the last column of Table C.1, and $\sigma = 0.5$, $\rho = 0$ and 0.3, and $\alpha = 0.5$.

Table C.1. Ranges of parameters used in numerical analysis

α	σ	ρ	$\| \mu_{z1} - \mu_{z2} \|$
0.00	0.20	0.00	0.00
0.50	0.34	0.09	0.42
1.00	0.49	0.18	0.88
1.50	0.63	0.27	1.30
2.00	0.77	0.36	1.75
2.50	0.92	0.45	2.18
3.00	1.06	0.54	2.60
	1.20	0.63	3.05
	1.35	0.72	3.48
	1.49	0.81	3.90
	1.63	0.90	4.36
			4.78
			5.23
			5.66

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